Monotone Perfection

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Abstract

Monotone equilibria may be undesirable in Bayesian games in the sense that players adopt weakly dominated strategies. To account for the possibility that the players might choose unintended strategies through a trembling hand, we propose an equilibrium refinement called "perfect monotone equilibrium." This notion strengthens the notion of monotone equilibrium in the sense that it satisfies the important property of admissibility in Bayesian games with finitely many actions, and the property of limit undominatedness in Bayesian games with infinitely many actions.

In a general class of Bayesian games where each player's action set is a sublattice of multi-dimensional Euclidean space and players' types are also multi-dimensional, a perfect monotone equilibrium is shown to exist under the supermodularity and increasing differences conditions. These conditions model the scenarios in which, informally, players' payoffs have complementarity in own actions and monotone incremental returns in own types. We demonstrate that the increasing differences condition is sharp by providing a two-player game that satisfies the widely adopted single crossing condition in the literature, but does not possess any perfect monotone equilibrium. To show the usefulness of our result in the setting with discontinuous payoffs, we provide various illustrative applications, including first-price auctions, all-pay auctions, and Bertrand competitions. Our result can be further extended to the setting with more general action spaces and type spaces.

Keywords: Incomplete information game, perfect monotone equilibrium, increasing differences, supermodularity, discontinuous payoffs

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1 Introduction

As a foundational element of game theory, the model of Bayesian game has provided a standard analytical tool and found applications across a wide range of fields. Auctions, for instance, are among the most successful applications of this model. In previous works on auctions, the focus is often on pure strategy monotone equilibrium, which suggests that bidders with higher valuations tend to submit higher bids. The existence of pure strategy monotone equilibrium has been established in Bayesian games with great success. The seminal work of Athey (2001) first established the existence of a pure strategy monotone equilibrium based on the single crossing condition. McAdams (2003) generalized this result to settings with multidimensional and partially ordered type and action spaces. Reny (2011) discovered contractibility to be automatically satisfied given any nonempty monotone best responses, and employed a powerful fixed-point theorem to establish the general existence of pure strategy monotone equilibria.[1]

While the above approach has proven useful, it can sometimes yield monotone equilibria that are undesirable. Consider an asymmetric first-price auction with two bidders. Bidder 1's private value v_1 is uniformly drawn from $[0, 5]$, and bidder 2's value v_2 is uniformly drawn from $[7, 8]$. Each bidder i submits a bid $b_i \in \{0, 1, 2, \cdots, 8\}$ after observing her value v_i. The bidder who submits a higher bid wins the good, with ties broken randomly. Here, $\hat{b}_1 \equiv 5$ and $\hat{b}_2 \equiv 6$ is a monotone equilibrium. To see it, given that bidder 2 bids 6, bidder 1's payoff is at most 0, achievable by bidding any number lower than 6. When bidder 1 bids 5, the best response of bidder 2 is to bid 6 for any v_2. Thus, $(\hat{b}_1, \hat{b}_2)$ is an equilibrium, which is trivially monotone. However, it is obvious that bidding 5 is weakly dominated by bidding 0 for bidder 1, as her payoff is at most 0 with a bid of 5 and at least 0 with a bid of 0. When bidder 2 bids 0, bidder 1 is strictly better off by bidding 0. The equilibrium $\hat{b}_1 \equiv 5$ and $\hat{b}_2 \equiv 6$ is unappealing because it involves weakly dominated actions.

The main purpose of this paper is to study an equilibrium concept called "perfect monotone equilibrium," which accounts for the possibility that the players might choose unintended strategies through a trembling hand, albeit with negligible probability. A perfect monotone equilibrium strengthens the notion of monotone equilibrium in the following sense: it satisfies the important property of admissibility in Bayesian games with finitely many actions, and the property of limit undominatedness in Bayesian games with infinitely many actions.

In Theorem 1, we prove the existence of a perfect monotone equilibrium in a class of Bayesian games, where each player's action set forms a sublattice of a multidimensional Euclidean space, and their types are multidimensional and atomless. Our equilibrium existence result relies on two widely adopted assumptions: each player's interim expected payoff (1) is supermodular in her own actions; and (2) has increasing differences in her own actions and types when others adopt monotone strategies. These two assumptions imply

[1] For more applications and developments of monotone equilibria, see, for example, Reny and Zamir (2004), McAdams (2006), and Prokopovych and Yannelis (2017, 2019).

that each player's interim payoff satisfies complementarity in own actions and monotone incremental returns in own types.

To establish the existence of a perfect monotone equilibrium in a Bayesian game, we construct a sequence of perturbed games that differ from the limit game in payoff functions. Importantly, every action in any perturbed game can be seen as a completely mixed action in the limit game. To obtain a perfect monotone equilibrium in the limit game, we need to show that each perturbed game retains certain properties from the limit game so that a monotone equilibrium exists in the perturbed game. As actions in the perturbed game are interpreted as completely mixed actions in the limit game, those properties must persist under expectations. The increasing differences condition is one such cardinal condition, which is stronger than the single crossing condition (SCC) in Athey (2001), McAdams (2003) and Reny (2011).[2] As remarked by Milgrom and Shannon (1994), the single crossing condition is not easy to work with, and they provided characterizations for the increasing differences condition. In particular, it is well known that the single crossing condition is an ordinal property that might not hold under expectations.[3] To demonstrate that the increasing differences condition is tight, we provide a counterexample of a two-player Bayesian game in Section 5.1. This example satisfies the single crossing condition, while the increasing differences condition fails only for one player, and a perfect monotone equilibrium fails to exist.

In several important applications of Bayesian games, players' payoffs are often discontinuous. For example, in first-price auctions for a single object, a bidder experiences a discrete change in payoffs when her bid shifts from being below opponents' highest bid to above it. Similarly, in price competitions, a firm's market share can jump significantly if its price slightly undercuts the current market price. Due to the payoff discontinuities, Theorem 1 does not apply directly to these environments. In Section 4, we offer three illustrative economic applications with affiliated types and a continuum of actions – first-price auction, all-pay auction, and Bertrand competition – to demonstrate how our results can be used to establish the existence of perfect monotone equilibria in Bayesian games with discontinuous payoffs.

Finally, we provide two extensions. In the first extension, we consider the setting with one-dimensional action spaces and type spaces as in Athey (2001). When a game has independent private values, it is shown that a perfect monotone equilibrium exists under the weaker single crossing condition. In the second extension, we extend our Theorem 1 to general Bayesian games as in Reny (2011), where the action spaces are compact locally complete metric semilattices and the type spaces are partially ordered probability spaces. We show that the equilibrium existence result continues to hold in this more general setting.

Our paper is related to the literature that provides equilibrium refinement aiming to eliminate undesirable equilibria. The classic work of Selten (1975) proposed the notion of

[2]The condition of supermodularity automatically holds in the single-dimensional setting as in Athey (2001).
[3]For further discussions, see Quah and Strulovici (2012) among others.

perfect equilibrium by introducing completely mixed strategies. Simon and Stinchcombe (1995) studied the possible issues with the notion of perfect equilibrium in strategic form games with compact action spaces. They discussed two essentially different approaches and investigated the relations among the various solution concepts. Bajoori, Flesch and Vermeulen (2013) examined the two approaches proposed in Simon and Stinchcombe (1995) in further details. Bajoori, Flesch and Vermeulen (2016) considered the notion of perfect equilibrium and discussed the properties of admissibility and limit undominatedness in Bayesian games.[4] All those papers do not invoke the monotone method, while our paper focuses on monotone strategies in Bayesian games and studies the notion of perfect monotone equilibrium.

There is another stream of literature on Bayesian games with strategic complementarity, which focuses on the existence of pure strategy equilibria and monotone comparative statics in supermodular games; see, for example, Milgrom and Roberts (1990), Vives (1990) and Milgrom and Shannon (1994). However, in these works, the strategies themselves need not be monotone in types.[5] In the current paper, we develop monotone methods applying to Bayesian games that may fail to exhibit complementarity across actions, but satisfy the monotone incremental returns in own type when others adopt monotone strategies.

This paper is also related to the literature that aims to provide sufficient conditions for the existence of equilibria in Bayesian games. Radner and Rosenthal (1982) worked with the conditions of independent atomless types and private values. Milgrom and Weber (1985) allowed for payoffs with private values and correlations among the players by working with conditionally independent types. He and Yannelis (2016) and Carbonell-Nicolau and McLean (2018) studied Bayesian games with discontinuous payoffs. Fu and Yu (2018) provided sufficient conditions that ensure the existence of Pareto-undominated and socially-maximal pure strategy equilibria. He and Sun (2019) introduced a general condition called "coarser inter-player information" and showed that it is necessary and sufficient for the existence of pure strategy equilibria. All those papers do not consider monotone strategies or perfect equilibria.

The remainder of the paper is organized as follows. Section 2 introduces the model of Bayesian games and the notion of perfect monotone equilibrium. Section 3 presents the key condition and proves the existence of perfect monotone equilibrium. In Section 4, we provide applications in auctions and price competitions. Section 5 includes further examples. In Section 6, we discuss one extension in the independent private value setting and another extension in the setting with general action and type spaces. The proofs are left in Appendix.

[4] The notion of perfect equilibrium has also been studied in other general environments; see, for example, Carbonell-Nicolau (2011) for discontinuous games, Carbonell-Nicolau and McLean (2014) for potential games, and Rath (1994, 1998) and Sun and Zeng (2020) for large games.

[5] It was further shown in Van Zandt and Vives (2007) that in the class of monotone supermodular games the extremal pure strategy equilibria are monotone in types. An algorithm is provided to compute those equilibria.

2 Bayesian Games

In this section, we examine a class of general Bayesian games, where each player's action set forms a sublattice of a multidimensional Euclidean space, and their types are multidimensional and atomless.

2.1 Model

Before introducing the model, we first introduce the concept of lattices. Let $(L, \geq)$ be a partially ordered set. For any $S \subseteq L$, let $\vee S$ denote the least upper bound of S in L (if it exists), satisfying

- $\vee S \geq s$ for any $s \in S$, and
- for $c \in L$, if $c \geq s$ for any $s \in S$, then $c \geq \vee S$.

Similarly, let $\wedge S$ denote the greatest lower bound of S in L (if it exists), satisfying

- $\wedge S \leq s$, for all $s \in S$,
- for all $c \in L$, if $c \leq s$ for all $s \in S$, then $c \leq \wedge S$.

For example, if $S = \{a, b\}$, then $a \vee b = \vee S$ and $a \wedge b = \wedge S$.

Definition 1. *1. A lattice is a partially ordered set $(L, \geq)$ such that for all $a, b \in L$, $a \vee b \in L$ and $a \wedge b \in L$.*

2. A subset E of L is a sublattice if E forms a lattice; that is, for any $a, b \in E$, $a \vee b \in E$ and $a \wedge b \in E$.

3. A sublattice E is complete if and only if for all $S \subseteq E$, $\vee S \in E$ and $\wedge S \in E$. Every finite sublattice is complete.

Now we are ready to present the formal model of Bayesian games.

- The set of players is denoted by $I = \{1, 2, \ldots, n\}$, $n \geq 2$.
- For each $i \in I$, the action space of player i is A_i, which is a complete sublattice in some Euclidean space $\mathbb{R}^s$ with respect to the product order.[6] Denote $A = \prod_{i=1}^n A_i$.
- Player i's type t_i is drawn from the type space $T_i = [0, 1]^l$ for some $l \in \mathbb{Z}_+$.[7] Denote $T = \prod_{i=1}^n T_i$. The joint density function $f\colon T \to \mathbb{R}_+$ on types is bounded above by $\overline{M}$ and bounded below by $\underline{M} > 0$. The type space is endowed with the product order and the usual Euclidean topology. The marginal distribution on player i's type space is denoted by λ_i; that is, for any Borel set $B_i \subseteq T_i$,

$$\lambda_i(B_i) = \int_{T_{-i}} \int_{B_i} f(t_i | t_{-i}) \, \mathrm{d}t_i \, \mathrm{d}t_{-i}.$$

[6] For any two elements $x, y \in A_i$, $x = (x_1, x_2, \ldots, x_s)$ and $y = (y_1, y_2, \ldots, y_s)$, $x \geq y$ in the product order if and only if $x_j \geq y_j$ for $j \in \{1, 2, \ldots, s\}$, and $x > y$ if and only if $x \geq y$ and $x \neq y$.

[7] The assumption of a common type space is made purely for simplicity. All results can be extended to the setting where players have different type spaces of varying dimensionality.

- Given action profile $a \in A$ and type profile $t \in T$, the payoff of player i is $u_i(a, t)$, which is bounded, jointly measurable, and continuous in a.
- For each player $i \in I$, a behavioral strategy (resp. pure strategy) is a measurable mapping from T_i to $\mathcal{M}(A_i)$ (resp. A_i), where $\mathcal{M}(A_i)$ is the set of probability measures on A_i.
- A monotone (or increasing) strategy α_i is a pure strategy increasing in player i's types; that is, $\alpha_i(t_i^H) \geq \alpha_i(t_i^L)$ for $t_i^H \geq t_i^L$. For each $i \in I$, let $\mathcal{F}_i$ be the set of monotone strategies of player i. As usual, $\mathcal{F} = \prod_{i=1}^n \mathcal{F}_i$ and $\mathcal{F}_{-i} = \prod_{j \neq i, j \in I} \mathcal{F}_j$.
- Given a strategy profile $g = (g_1, \ldots, g_n)$, the interim payoff of player i depends on her own action a_i, own type t_i, and other players' strategies g_{-i},

$$V_i(a_i, t_i; g_{-i}) = \int_{T_{-i}} \int_{A_{-i}} u_i(a_i, a_{-i}, t_i, t_{-i}) \underset{j \in I, j \neq i}{\otimes} g_j(t_j, \mathrm{d}a_j) f(t_{-i}|t_i) \, \mathrm{d}t_{-i}.$$

  Player i's expected payoff is

$$U_i(g) = \int_{T_i} \int_{A_i} V_i(a_i, t_i; g_{-i}) g_i(t_i, \mathrm{d}a_i) \lambda_i(\mathrm{d}t_i)$$

  where λ_i is the marginal on T_i.

Let $\mathrm{BR}_i(t_i, g_{-i})$ be the collection of best responses to a strategy profile g_{-i} at player i's type t_i, and $\mathrm{BR}_i(g_{-i})$ be the collection of player i's best responses to a strategy profile g_{-i}.

Definition 2. *1. A Bayesian Nash equilibrium is a strategy profile $g^* = (g_1^*, g_2^*, \ldots, g_n^*)$ such that $g_i^* \in BR_i(g_{-i}^*)$ for every player $i \in I$. Moreover, if $g^* = (g_1^*, g_2^*, \ldots, g_n^*)$ is a pure strategy profile, then g^* is a pure strategy Bayesian Nash equilibrium.*

2. A monotone equilibrium is a pure strategy Bayesian Nash equilibrium $g^* = (g_1^*, g_2^*, \ldots, g_n^*)$ with g_i^* being an increasing strategy for each player $i \in I$.

2.2 Perfect monotone equilibrium

Throughout this paper, we will focus on monotone equilibria. However, as demonstrated via the illustrative example in the introduction, monotone equilibria do not rule out the possibility that players might choose weakly dominated strategies. To address this issue, we strengthen the notion of monotone equilibrium by requiring that such equilibria be perfect, which is one of the most commonly used refinement of Nash equilibria.

A strategy profile $g = (g_1, \ldots, g_n)$ is called a completely mixed strategy profile if for each player i, each $t_i \in T_i$, and any nonempty open subset O_i of A_i, $g_i(t_i, O_i) > 0$.

Definition 3 (Monotone Perfection)**.** *a. A strategy profile $g = (g_1, g_2, \cdots, g_n)$ is said to be perfect, if there exists a sequence of completely mixed strategy profiles $\{g^k = (g_1^k, g_2^k, \cdots, g_n^k)\}_{k=1}^{\infty}$ such that for every player i and λ_i-almost all t_i, the following properties hold.*

(1). $\lim_{k\to\infty} \rho\big(g_i^k(t_i,\cdot), \mathrm{BR}_i(t_i, g_{-i}^k)\big) = \lim_{k\to\infty} \inf_{\sigma_i \in \mathrm{BR}_i(t_i, g_{-i}^k)} \rho(g_i^k(t_i,\cdot), \sigma_i) = 0.$[8]

(2). $\lim_{k\to\infty} \rho\big(g_i^k(t_i,\cdot), g_i(t_i,\cdot)\big) = 0.$

b. *A strategy profile g is called a perfect equilibrium if g is both perfect and a Bayesian Nash equilibrium.*

c. *A strategy profile g is a perfect monotone equilibrium if it is both perfect and a monotone equilibrium.*

The above notion of perfect equilibrium (*i.e.*, (a) and (b)) is standard in the literature; see Selten (1975) and Simon and Stinchcombe (1995) for normal-form games and Bajoori, Flesch and Vermeulen (2016) for Bayesian games. We strengthen this notion in (c) by requiring the equilibrium be monotone.

3 Main Results

In this section, we provide sufficient conditions to guarantee the existence of perfect monotone equilibria. In particular, it is shown that when the other players adopt monotone strategies, if player i's payoff satisfies the conditions of increasing differences and supermodularity for each $i \in I$, then a Bayesian game possesses a perfect monotone equilibrium.

Definition 4 (Supermodularity). *Let $(X, \geq, \vee, \wedge)$ be a lattice, and Θ an index set. A function $h\colon X \times \Theta \to \mathbb{R}$ is supermodular in x (or SPM(x)) if and only if for any $x, x' \in X$ and $\theta \in \Theta$,*

$$h(x \vee x', \theta) + h(x \wedge x', \theta) \geq h(x, \theta) + h(x', \theta).$$

Definition 5 (Increasing differences). *Let $(X, \geq, \vee, \wedge)$ be a lattice, $(Y, \geq)$ a partially ordered set, and Θ an index set. A function $h\colon X \times Y \times \Theta \to \mathbb{R}$ satisfies increasing differences condition (IDC) in $(x, y) \in X \times Y$ if and only if for any $x' > x$, $y' > y$, and $\theta \in \Theta$,*

$$h(x', y', \theta) - h(x, y', \theta) \geq h(x', y, \theta) - h(x, y, \theta).$$

Assumption 1. *For each $i \in I$ and monotone strategies $g_{-i} \in \mathcal{F}_{-i}$, player i's interim payoff $V_i(a_i, t_i; g_{-i}(\cdot))$*

1. *is supermodular in a_i for any $t_i \in T_i$, and*
2. *has increasing differences in (a_i, t_i).*

Remark 1.
- *If $A_i \subseteq \mathbb{R}$, then Assumption 1 (1) is trivially satisfied.*

[8] Here ρ is the Prohorov metric: for $\nu, \mu \in \mathcal{M}(A_i)$,

$$\rho(\nu, \mu) = \inf\{\epsilon\colon \nu(B) \leq \mu(B^\epsilon) + \epsilon \text{ and } \mu(B) \leq \nu(B^\epsilon) + \epsilon \text{ for any Borel set } B \subset A_i\},$$

where d is the Euclidean metric, $B^\epsilon = \{b \in A_i\colon d(b, B) < \epsilon\}$ and $d(b, B) = \inf_{b' \in B} d(b, b')$.

- *Assumption 1 is slightly weaker than assuming that $V_i(a_i, t_i; g_{-i}(\cdot))$ is supermodular in (a_i, t_i). To see it, note that given g_{-i}, if $V_i(a_i, t_i; g_{-i}(\cdot))$ is supermodular in (a_i, t_i), then it is clear that V_i has increasing differences in (a_i, t_i); but the converse is not true.*

The following theorem presents the existence result of perfect monotone equilibrium.

Theorem 1. *A perfect monotone equilibrium exists under Assumption 1.*

Remark 2. *The notion of perfect monotone equilibrium refines the notion of monotone equilibrium. The existence of monotone equilibria has been extensively studied in the literature. Athey (2001) provided sufficient conditions for the existence of a monotone equilibrium in the one-dimensional setting. McAdams (2003) obtained the equilibrium existence in settings with multidimensional actions and multidimensional types. Reny (2011) established the existence of a monotone equilibrium in a very general setting. These papers work with the (weak) single crossing condition,*[9] *which is slightly weaker than the condition of increasing differences. McAdams (2003) further assumed quasi-supermodularity for interim payoff function V_i. Note that the supermodularity condition is automatically satisfied in Athey (2001), which focuses on the single dimensional setting.*

Topkis (1978) indicated that supermodularity and the increasing differences condition can be easily characterized by smooth functions. Milgrom and Shannon (1994) pointed out that quasi-supermodularity and the single crossing condition may seem abstract and not easy to check, and instead provided characterizations for supermodularity and increasing differences condition. In Section 5.1, we provide an example to demonstrate that the increasing differences condition is sharp in the general setting. In this example, (1) players have single dimensional independent types and interdependent payoffs; (2) the single crossing condition holds for all the players; (3) the increasing differences condition fails for one player; but (4) there does not exist a perfect monotone equilibrium.

In Section 6.1 below, we study the specific IPV setting; that is, Bayesian games with independent private values. We show that in this classic setting, the existence of a perfect monotone equilibrium can be obtained under the single crossing condition.

Our result can be extended to the more general setting in which the action spaces are compact metric spaces and the type spaces are general measure spaces. We choose to work with the Euclidean space in Theorem 1 for simplicity. In Section 6.2, we show that the above equilibrium existence result still holds in the general environment as in Reny (2011) under our Assumption 1.

Remark 3 (The key of the proof)**.** *To prove the existence of a perfect monotone equilibrium in the game G, we construct a sequence of games $\{G^m\}_{m=1}^{\infty}$ converging to G. The key of the construction is summarized below.*

[9] Given g_{-i}, player i's interim payoff satisfies the single crossing condition in Milgrom and Shannon (1994) if for $a_i^H > a_i^L$ and $t_i^H > t_i^L$, $V_i(a_i^H, t_i^L, g_{-i}(\cdot)) - V_i(a_i^L, t_i^L, g_{-i}(\cdot)) \geq (>) 0$ implies that $V_i(a_i^H, t_i^H, g_{-i}(\cdot)) - V_i(a_i^L, t_i^H, g_{-i}(\cdot)) \geq (>) 0$. Reny (2011) adopted a weaker version of this condition.

- *Each G^m is a slight perturbation of G in the following sense. The game G^m differs from the limit game G only in its payoff functions: when players play the action profile a at type profile t, player i's payoff $u_i^m(a,t)$ in G^m equals $u_i(a^m,t)$, where a_j^m is a completely mixed strategy putting probability $1-\frac{1}{m}$ at the action a_j for each player j.*
- *It is shown that each game G^m satisfies Assumption 1 above, and has a monotone equilibrium g^m. Importantly, g^m can be interpreted as a completely mixed strategy $\overline{g}^m$ in the game G such that $\overline{g}_i^m$ is an approximate best response of $\overline{g}_{-i}^m$ for each player i. The sequence $\{g^m\}$ has a convergent subsequence $\{g^{m_k}\}_{k=1}^{\infty}$ with the limit g. We show that g is a perfect monotone equilibrium in the original game G.*

By interpreting g^m as a completely mixed strategy $\overline{g}^m$ in the game G, the expected payoff based on $\overline{g}^m$ inherits the increasing differences condition, which is a cardinal property. On the other hand, it is well known that the single crossing condition is an ordinal property, which often fails to hold when taking expectations.

If the action set A_i is finite for each $i \in I$, then it is well known that any perfect equilibrium satisfies the desirable feature of admissibility; that is, players do not play weakly dominated strategies.[10] However, if the action sets are not finite, then an admissible perfect equilibrium may fail to exist. This issue has been demonstrated in Simon and Stinchcombe (1995, Example 2.1), and still appears even if one considers monotone equilibria. In Example 4 below, we provide a countable-action Bayesian game with an inadmissible perfect equilibrium. To deal with this issue, in the infinite-action setting we adopt the notion of limit undominatedness from Bajoori, Flesch and Vermeulen (2016).

For each player $i \in I$, an action $a_i \in A_i$ is said to be weakly dominated if there exists a probability measure $\sigma_i \in \mathcal{M}(A_i)$ such that for λ_i-almost all t_i,

(1). $E_{g_{-i}}(u_i|t_i,a_i) \leq E_{g_{-i}}(u_i|t_i,\sigma_i)$ for any strategy profile g_{-i}; and

(2). there exists a strategy profile $\hat{g}_{-i}$ such that $E_{\hat{g}_{-i}}(u_i|t_i,a_i) < E_{\hat{g}_{-i}}(u_i|t_i,\sigma_i)$.

A probability measure $\sigma_i \in \mathcal{M}(A_i)$ is said to be undominated for a type t_i if there is no probability measure in $\mathcal{M}(A_i)$ that weakly dominates σ_i.

Definition 6. *1. **Admissibility**. A strategy profile g is said to be admissible if the induced action distribution $\int_{T_i} g_i(t_i)\lambda_i(\mathrm{d}t_i)$ puts zero probability on any weakly dominated action for each player $i \in I$.*

*2. **Limit undominatedness**. A strategy g_i is called limit undominated if there is a measurable set $S_i \subseteq T_i$ with $\lambda_i(S_i) = 0$ such that for any $t_i \in T_i \backslash S_i$, there is a sequence of undominated probability measures $\{\sigma_i^k\}_{k\geq 1}$ on the action space A_i for which $\rho(\sigma_i^k, g_i(t_i,\cdot)) \to 0$ as $k \to \infty$. A strategy profile $g = (g_1, g_2, \cdots, g_n)$ is called limit undominated if g_i is limit undominated for every player i.*

Corollary 1. *1. If A_i is finite for each $i \in I$, then any perfect monotone equilibrium is admissible.*

[10] For completeness, we formalize this statement as Lemma 1 and provide a proof in Appendix.

2. *If A_i is a complete sublattice in $\mathbb{R}^s$, then any perfect monotone equilibrium is limit undominated.*

4 Discontinuous Bayesian Games

In many important applications of Bayesian games, such as auctions and price competitions, it is natural to consider monotone equilibrium as players with higher valuations/costs are inclined to submit higher bids/charge higher prices. However, Theorem 1 cannot be directly applied to those applications, because the payoffs often exhibit discontinuity when a tie occurs. In this section, we shall demonstrate how to apply our result to show the existence of perfect monotone equilibria in various economic settings with discontinuous payoffs, such as first-price auctions, all-pay auctions, and Bertrand competitions.

First-price auctions

Consider the following first-price single-unit auction with affiliated types. There are $n \geq 2$ bidders. The value of bidder i is $v_i \in [0,1]$. The joint density for bidders' values is $f\colon [0,1]^n \to \mathbb{R}_+$. Players have affiliated types. That is, for any value profiles v and v',

$$f(v \vee v') \cdot f(v \wedge v') \geq f(v) \cdot f(v').$$

After receiving their values, each bidder i submits a bid b_i from $\{Q\} \cup [\underline{b}_i, \overline{b}_i]$, where $Q < \underline{b}_i \leq 0$ and Q corresponds to not participating in the auction. The bidder submitting the highest bid above Q wins the object, with tie-breaking done randomly and uniformly. If bidder i wins the object, then her payoff is $v_i - b_i$. Otherwise, bidder i receives payoff 0. To be precise, the payoff of bidder i is

$$u_i(b,v) = \begin{cases} \frac{1}{\#|I^w|}(v_i - b_i), & \text{if } b_i = \max_{j \in I} b_j > Q, \\ 0, & \text{otherwise,} \end{cases}$$

where $I^w = \{i \in I\colon b_i = \max\limits_{j \in I} b_j > Q\}$ is the set of bidders submitting the highest bid above Q.

Proposition 1. *A perfect monotone equilibrium exists in first-price auctions with affiliated types.*

All-pay auctions

Next, we consider an all-pay auction game with interdependent payoffs and affiliated types. Each bidder i receives a private signal $t_i \in T_i = [0,1]$. The joint density is $f\colon [0,1]^n \to \mathbb{R}_+$. After receiving their signals, each bidder i submits a bid from $B_i = \{Q\} \cup [\underline{b}_i, \overline{b}_i]$, where the quit option $Q < \underline{b}_i$ for all i.

The bidder submitting the highest bid above Q wins the object, with tie-breaking done randomly and uniformly. All the bidders who bid above Q need to pay their bids. If bidder i wins the object at bid b_i, then her payoff is given by $w_i(b_i, t) - b_i$; otherwise, bidder i receives payoff $-b_i$. If bidder i quits the game, then she receives payoff 0. In particular, bidder i's payoff is

$$u_i(b,t) = \begin{cases} \frac{1}{\#|I^w|} w_i(b_i,t) - b_i, & \text{if } b_i = \max_{j \in I} b_j > Q; \\ 0, & \text{if } b_i = Q; \\ -b_i, & \text{otherwise.} \end{cases}$$

We make the following assumption on the payoff functions.

Assumption 2. 1. *The payoff function w_i is bounded and measurable on $[\underline{b}_i, \overline{b}_i] \times [0,1]^n$, and continuous in b_i for each $t \in [0,1]^n$.*

2. *The function $w_i(b_i, t)$ is increasing in t_{-i} and strictly increasing in t_i.*

3. *The difference $w_i(b_i^H, t) - w_i(b_i^L, t)$ is increasing in t whenever $b_i^H > b_i^L \geq \underline{b}_i$.*

Proposition 2. *Under Assumption 2, an all-pay auction with affiliated types possesses a perfect monotone equilibrium.*

Bertrand competitions with unknown costs

The third application is a price competition game in which firms have private costs. There are n firms that compete by setting prices $p_i \in [0, \overline{p}_i]$, where $\overline{p}_i \geq 1$ for each i. Each firm i knows its marginal cost $c_i \in [0,1]$. The joint density is $f\colon [0,1]^n \to \mathbb{R}_+$. The market demand is given by $D(p)$, where $p = (p_1, p_2, \cdots, p_n)$, and $D(p)$ is continuous in p.

If $p_i = \min_{1 \leq j \leq n} p_j$, then $D_i(p) = \frac{D(p)}{\#\{j : p_j = p_i\}}$. Otherwise, $D_i(p) = 0$. The demand function $D_i(p)$ is increasing in p_{-i} and decreasing in p_i. Firm i's profit is

$$u_i(c_i, p_i, p_{-i}) = (p_i - c_i) D_i(p).$$

Proposition 3. *The Bertrand competition with affiliated unknown costs has a perfect monotone equilibrium.*

Remark 4. *Given a Bayesian game G with discontinuous payoffs, the proofs for the propositions above proceed as follows. We first repeat the argument in Remark 3 to construct a sequence of perturbed games $\{G^m\}$, and then further discretize G^m to $\{G^{mk}\}_{k \geq 1}$ for each $m \geq 1$.*

1. *In each G^{mk}, a monotone equilibrium g^{mk} exists.*

2. *Taking $k \to \infty$, we get a monotone strategy g^m in G^m, which can be interpreted as a completely mixed strategy in G.*

3. *Taking $m \to \infty$, we get a monotone strategy g in G. The aim is to show that g is an equilibrium in the original game G.*

The difficulty in this double-limit approach is that g^m may not be an equilibrium in G^m in the second step. In the literature, to make sure that the limit monotone strategy g^m is an equilibrium in G^m when invoking this kind of asymptotic argument, a key step is to show that player i's interim payoff at (almost all) type t_i by taking action $g_i^m(t_i)$ against g_{-i}^m is nonnegative, as otherwise player i can choose to quit (see, for example, Reny and Zamir (2004)). In our game G^m, as g^m is interpreted as a completely mixed strategy in G, player i's payoff induced by g^m is written as the summation of countably many payoff terms. This summation is still nonnegative. However, to adopt the standard limit argument in the literature, we need every term in this summation to be nonnegative, which may not be true. We show that this issue can be addressed in a large class of applications.

5 Examples

5.1 A Bayesian game satisfying SCC but has no perfect monotone equilibrium

Below, we provide a simple Bayesian game, in which there are only two players and each player has binary actions. We shall verify that each player's interim payoff satisfies the single crossing condition when opponents adopt monotone strategies, and only player 1's payoff violates the increasing differences condition. This game has a monotone equilibrium, but has no perfect monotone equilibrium. It demonstrates that the increasing differences condition is in general sharp.

Example 1. *There are two players $\{1, 2\}$. Their types are independently and uniformly drawn from the unit interval $[0, 1]$. Let t_i denote player i's type. Both players have binary actions $\{1, 2\}$. The payoff tables are given below.*

Player 1 \ Player 2	1	2
1	$\frac{5}{2} - \frac{4}{3}t_1$	$\frac{1}{2} + \frac{1}{2}t_2$
2	$\frac{55}{24} - \frac{t_2}{6} - \frac{7}{6}t_1$	$2 + t_2 - \frac{7}{6}t_1$

Table 1: Player 1's payoff

Player 2 \ Player 1	1	2
1	-1	7
2	1	-1

Table 2: Player 2's payoff

Note that player 2's payoff does not depend on types, and thus the increasing differences condition holds. The supermodularity condition is trivially satisfied as both players have single dimensional action spaces.

Claim 1. *1. The game satisfies the single crossing condition, but not the increasing differences condition.*

2. *This game has a unique monotone equilibrium.*

3. *This game does not possess any perfect monotone equilibrium.*

5.2 Two auction games with both perfect and imperfect monotone equilibria

In this section, we provide two examples of auction games. Both examples possess imperfect monotone equilibria. It will be clear that the examples satisfy Assumption 1. Thus, both examples also have perfect monotone equilibria. The first example revisits the motivating example of the first-price auction in the introduction. The second example is a second-price auction in the IPV setting.

Example 2 (First-price auction)**.** *There are two bidders. Bidders* 1*'s valuation* v_1 *is uniformly drawn from the interval* $[0, 5]$*, and bidder* 2*'s valuation* v_2 *is uniformly drawn from the interval* $[7, 8]$*. The action spaces for both bidders are* $\{0, 1, 2, \cdots, 8\}$*. The bidder who submits a higher bid wins the good and pays his bid. If they offer the same bid amount, then the two bidders break the tie by flipping a coin.*

In the introduction, we have provided a monotone equilibrium that is imperfect. Below, we provide a perfect monotone equilibrium for this first-price auction.

Claim 2. *The strategy profile*

$$b_1(v_1) = \begin{cases} 0 & v_1 \in [0, \frac{3}{2}), \\ 1 & v_1 \in [\frac{3}{2}, 3), \\ 3 & v_1 \in [3, 5]; \end{cases} \qquad \text{and} \qquad b_2(v_2) \equiv 3$$

is a perfect monotone equilibrium.

In the following, we provide another example of a second-price auction.

Example 3 (Second-price auction)**.** *There are two bidders. Bidders* 1 *and* 2*'s valuations* v_1 *and* v_2 *are uniformly drawn from the interval* $[1, 2]$*. The action spaces for both bidders are* $\{0, 1, 2\}$*. The bidder who offers a higher bid wins the good and pays another bidder's bid. If they offer the same bid amount, then they win the good by flipping a coin.*

It is clear that $\hat{b}_1 \equiv 0$ and $\hat{b}_2 \equiv 2$ forms a monotone equilibrium. Given that bidder 2 always bids 2, bidder 1's payoff is at most 0. Conversely, if bidder 1 always bids 0, then bidder 2's highest payoff is v_2, achievable by bidding 2. However, bidding 0 is weakly dominated by bidding 1 for bidder 1. In particular, the payoff of bidder 1 when he submits a bid 0 is always less than or equal to his payoff when he submits a bid 1. When bidder 2 submits 0, bidder 1's payoff strictly increases by bidding 1. Therefore, $\hat{b}_1 \equiv 0$ and $\hat{b}_2 \equiv 2$ is an imperfect monotone equilibrium.

In the following, we provide a perfect monotone equilibrium for this second-price auction.

Claim 3. *The strategy profile*

$$b_i(v_i) = \begin{cases} 1 & v_i \in [1, \frac{3}{2}), \\ 2 & v_i \in [\frac{3}{2}, 2], \end{cases}$$

for $i \in \{1, 2\}$ is a symmetric perfect monotone equilibrium.

6 Extensions

In this section, we study two extensions of Theorem 1. The first extension focuses on the setting with one-dimensional action spaces and type spaces as in Athey (2001). We show that the single crossing condition is sufficient for the existence of a perfect monotone equilibrium in Bayesian games with independent private values. The second extension concerns with Bayesian games with general action spaces and type spaces as in Reny (2011). We generalize Theorem 1 to this more general environment.

6.1 Bayesian games with one-dimensional action spaces and type spaces

We focus on a class of Bayesian games with one-dimensional action spaces and type spaces. For simplicity, we shall only describe the differences from the model in Section 2.1.

- For each player $i \in I$, the action space is A_i, which is a compact subset of $\mathbb{R}$. Denote $\underline{a}_i \equiv \min A_i$ and $\overline{a}_i \equiv \max A_i$.
- The type space of player i is $T_i = [\underline{t}_i, \overline{t}_i]$.

Assumption 3 (IPV)**.** *For each $i \in I$,*

1. *player i has private values: $u_i(a, t) = u_i(a, t_i)$; and*
2. *types are independent: $f(t) = f_1(t_1) \ldots f_n(t_n)$, where f_i is the density for player i's type t_i.*

Assumption 4 (SCC)**.** *For each $i \in I$, player i's interim payoff $V_i(a_i, t_i; \alpha_{-i}(\cdot))$ satisfies the single crossing condition in (a_i, t_i) for any $\alpha_{-i} \in \mathcal{F}_{-i}$.*

Theorem 2. *Suppose that Assumptions 3 and 4 hold. For each player $i \in I$, if either A_i is finite, or $A_i = [\underline{a}_i, \overline{a}_i]$ and $u_i(a, t)$ is continuous in a, then there exists a perfect monotone equilibrium.*

Remark 5. *Recall the discussions in Remark 3. To show the existence of a perfect monotone equilibrium in a game G, we construct a sequence of games $\{G^m\}_{m=1}^{\infty}$ that converges to G, where each game G^m differs from G only in its payoff. By imposing the increasing differences condition on $V_i(a_i, t_i; g_{-i}(\cdot))$, we can show that $V_i^m(a_i, t_i; g_{-i}(\cdot))$ also satisfies the increasing differences condition. It further implies that $V_i^m(a_i, t_i; g_{-i}(\cdot))$ satisfies the single crossing condition. In the single-dimensional setting, one can apply Theorems 1 and 2 in Athey (2001) to obtain a monotone equilibrium in G^m. This argument would not work in general if $V_i(a_i, t_i; g_{-i}(\cdot))$ only satisfies the single crossing condition. However, in the IPV setting, it can be shown that SCC on the payoff V_i implies SCC on the payoff V_i^m. As a result, we are able to weaken the increasing differences condition to the single crossing condition and obtain Theorem 2.*

In Example 1 above, player 1 has an interdependent payoff that depends on both players' types. The interim payoff V_1 satisfies the single crossing condition, but not the increasing differences condition. As shown in Claim 1, there is no perfect monotone equilibrium.

6.2 Bayesian games with general action spaces and type spaces

Below, we follow Reny (2011) to study a class of Bayesian games with general action spaces and type spaces. The action spaces are compact locally complete metric semilattices and the type spaces are partially ordered probability spaces.

Let $(L, \geq)$ be a partially ordered set. If L is endowed with a sigma-algebra $\mathcal{L}$, then the partial order $\geq$ on L is called measurable if the set $\{(a, b) \in L \times L \colon a \geq b\} \in \mathcal{L} \otimes \mathcal{L}$. The partial order $\geq$ on L is called closed if $\{(a, b) \in L \times L \colon a \geq b\}$ is closed in the product topology on $L \times L$. The partial order $\geq$ is called convex if L is a subset of a real vector space and $\{(a, b) \in L \times L \colon a \geq b\}$ is convex.

We say that L is a semilattice if every pair of points in L has a least upper bound in L. A lattice is a semilattice. If L is a semilattice endowed with a metric d and the join operator $\vee$ is a continuous function from $L \times L$ to L, then L is a metric semilattice. A semilattice L is complete if and only if the least upper bound $\vee S \in L$ for every nonempty subset $S \subset L$. A metric semilattice L is locally complete if for every $a \in L$ and every neighbourhood U of a, there is a neighbourhood W of a contained in U such that every nonempty subset $S \subset W$ has a least upper bound $\vee S$ in W.

Next, we describe Bayesian games with general action spaces and type spaces.

- The player space is $I = \{1, 2, \ldots, n\}$, $n \geq 2$.
- For each $i \in I$, the action space of player i is A_i, which is a compact metric space and a semilattice with a closed partial order.
- Either (i) A_i is a convex subset of a locally convex topological vector space and the partial order on A_i is convex; or (ii) A_i is a locally complete metric semilattice. It is possible for (i) to hold for some players and (ii) to hold for others.

- The type space of player $i \in I$ is T_i endowed with the σ-algebra $\mathcal{T}_i$. The set T_i is partially ordered, and the partial order on T_i is measurable.
- The common prior over the players' type spaces is a countably additive probability measure λ on T. Let $(T_i, \mathcal{T}_i, \lambda_i)$ be an atomless probability space, where λ_i is the marginal of λ on T_i for each $i \in I$.
- There is a countable subset T_i^0 of T_i such that every set in $\mathcal{T}_i$ with positive probability under λ_i contains two points between which lies a point in T_i^0.[11]
- Given the action profile $a \in A$ and type profile $t \in T$, player i's payoff is $u_i(a, t)$, which is bounded, jointly measurable, and continuous in a for every $t \in T$.

Theorem 3. *Under Assumption 1, there exists a perfect monotone equilibrium.*

Remark 6. *To prove Theorem 3, we repeat the argument outlined in Remark 3. Recall that a sequence of games $\{G^m\}$ is carefully constructed to converge to the limit game G. Each game G^m also satisfies Assumption 1. Reny (2011) proved the existence of monotone equilibria by assuming that each player i's interim payoff satisfies weak quasi-supermodularity and weak single crossing condition. Our Assumption 1 is stronger than the weak quasi-supermodularity and weak single crossing condition. Thus, there exists a monotone equilibrium in G^m, which is interpreted as a completely mixed strategy in G. By (possibly) passing to a subsequence, we get a monotone equilibrium in the limit game G that is also perfect. As the argument is almost the same as the proof of Theorem 1, we omit it for simplicity.*

7 Conclusion

In this paper, we propose an equilibrium refinement called "perfect monotone equilibrium" to address the issue that players may choose weakly dominated strategies in monotone equilibria. In Bayesian games with finitely many actions, a perfect monotone equilibrium is admissible; in Bayesian games with infinitely many actions, a perfect monotone equilibrium is limit undominated.

In a general class of Bayesian games where each player's action set is a sublattice of multi-dimensional Euclidean space and players' types are also multi-dimensional, to prove the existence of a perfect monotone equilibrium, we make two widely-adopted assumptions: players' payoffs are supermodular in own actions and have increasing differences in own actions and types. These two assumptions imply complementarity in own actions and monotone incremental returns in own types. We demonstrate that our condition is sharp via counterexamples. To show the usefulness of our result in economic settings, we provide various illustrative applications, including first-price auctions, all-pay auctions, and Bertrand competitions. Our result can be further extended to the IPV setting as in

[11] For $a, b, c \in L$, we say that b lies between a and c if $a \geq b \geq c$.

Athey (2001), and the more general setting as in Reny (2011) where the action spaces are compact locally complete metric semilattices and the type spaces are partially ordered probability spaces.

References

Susan Athey, Single crossing properties and the existence of pure strategy equilibria in games of incomplete information, *Econometrica* **69** (2001), 861–889. 3, 4, 9, 15, 16, 18

Elnaz Bajoori, Jănos Flesch and Dries Vermeulen, Perfect equilibrium in games with compact action spaces, *Games and Economic Behavior* **82** (2013), 490–502. 5

Elnaz Bajoori, Jănos Flesch and Dries Vermeulen, Behavioral perfect equilibrium in Bayesian games, *Games and Economic Behavior* **98** (2016), 78–109. 5, 8, 10, 21

Oriol Carbonell-Nicolau, On the existence of pure-strategy perfect equilibrium in discontinuous games, *Games and Economic Behavior* **71** (2011), 23–48. 5

Oriol Carbonell-Nicolau and Richard P. McLean, Refinements of Nash equilibrium in potential games, *Theoretical Economics*, **9** (2014), 555–582. 5

Oriol Carbonell-Nicolau and Richard P. McLean, On the existence of Nash equilibrium in Bayesian games, *Mathematics of Operations Research* **43** (2018), 100–129. 5

Haifeng Fu and Haomiao Yu, Pareto refinements of pure-strategy equilibria in games with public and private information, *Journal of Mathematical Economics* **79** (2018), 18–26. 5

Wei He and Yeneng Sun, Pure-strategy equilibria in Bayesian games, *Journal of Economic Theory* **180** (2019), 11–49. 5

Wei He and Nicholas C. Yannelis, Existence of equilibria in discontinuous Bayesian games, *Journal of Economic Theory* **162** (2016), 181–194. 5

David McAdams, Isotone equilibrium in games of incomplete information, *Econometrica* **71** (2003), 1191–1214. 3, 4, 9

David McAdams, Monotone equilibrium in multi-unit auctions, *The Review of Economic Studies* **73** (2006), 1039–1056. 3

Paul R. Milgrom and John Roberts, Rationalizability, learning, and equilibrium in games with strategic complementarities, *Econometrica* **58** (1990), 1255–1277. 5

Paul R. Milgrom and Chris Shannon, Monotone comparative statics, *Econometrica* **62** (1994), 157–180. 4, 5, 9

Paul R. Milgrom and Robert J. Weber, A theory of auctions and competitive bidding, *Econometrica* **50** (1982), 1089–1122. 25, 28, 29, 33, 37

Paul R. Milgrom and Robert J. Weber, Distributional strategies for games with incomplete information, *Mathematics of Operations Research* **10** (1985), 619–632. 5

Pavlo Prokopovych and Nicholas C. Yannelis, On strategic complementarities in discontinuous games with totally ordered strategies, *Journal of Mathematical Economics* **70** (2017), 147–153. 3

Pavlo Prokopovych and Nicholas C. Yannelis, On monotone approximate and exact equilibria of an asymmetric first-price auction with affiliated private information, *Journal of Economic Theory* **184** (2019), 104925. 3

John K-H. Quah and Bruno Strulovici, Aggregating the single crossing property, *Econometrica* **80** (2012), 2333–2348. 4

Kali P. Rath, Some refinements of Nash equilibria of large games, *Games and Economic Behavior* **7** (1994), 92–103. 5

Kali P. Rath, Perfect and proper equilibria of large games, *Games and Economic Behavior* **22** (1998), 331–342. 5

Reinhard Selten, Reexamination of the perfectness concept for equilibrium points in extensive games, *International Journal of Game Theory* **4** (1975), 25–55. 4, 8

Leo K. Simon and Maxwell B. Stinchcombe, Equilibrium refinement for infinite normal-form games, *Econometrica* **63** (1995), 1421–1443. 5, 8, 10, 21

Xiang Sun and Yishu Zeng, Perfect and proper equilibria in large games, *Games and Economic Behavior* **119** (2020), 288–308. 5

Roy Radner and Robert W. Rosenthal, Private information and pure-strategy equilibria, *Mathematics of Operations Research* **7** (1982), 401–409. 5

Philip J. Reny, On the existence of monotone pure-strategy equilibria in Bayesian games, *Econometrica* **43** (2011), 449–553. 3, 4, 9, 15, 16, 17, 18, 20

Philip J. Reny and Shmuel Zamir, On the existence of pure strategy monotone equilibria in asymmetric first-price auctions, *Econometrica* **72** (2004), 1105–1125. 3, 13

Donald M. Topkis, Minimizing a submodular function on a lattice, *Operations Research* **26** (1978), 305–321. 9

Xavier Vives, Nash equilibrium with strategic complementarities, *Journal of Mathematical Economics* **19** (1990), 305–321. 5

Timothy Van Zandt and Xavier Vives, Monotone equilibria in Bayesian games of strategic complementarities, *Journal of Economic Theory* **134** (2007), 339–360. 5

8 Appendix

8.1 Proof of Theorem 1 and Corollary 1

We first present the proof of Theorem 1, organized into three distinct steps. In Step 1, we construct a sequence of Bayesian games, $\{G^m\}_{m=1}^{\infty}$, that converges to the limit Bayesian game G, where G^m differs from G only in its payoff functions. In Step 2, we demonstrate that each G^m possesses a monotone equilibrium g^m, and we establish the existence of a subsequence of $\{g^m\}_{m=1}^{\infty}$ that converges to an increasing strategy g. In Step 3, we show that g is a perfect equilibrium. Given that g is an increasing strategy, it follows that g is a monotone equilibrium and, therefore, a perfect monotone equilibrium, completing our proof.

Step 1. Since A_i is a complete sublattice, and thus a compact metric space, for each $i \in I$, there exists a countable dense subset $S_i \subseteq A_i$, where $S_i = \{s_i^k\}_{k=1}^{\infty}$. For each $a_i \in A_i$ and $m \in \mathbb{N}$, let a_i^m denote a completely mixed probability measure on A_i putting probability $1 - \frac{1}{m}$ on action a_i, where $a_i^m = (1-\frac{1}{m})\delta a_i + \frac{1}{m}\sum_{k=1}^{\infty}\frac{1}{2^k}\delta_{s_i^k}$. In game G^m, player i's payoff function is $u_i^m(a,t)$, and

$$u_i^m(a,t) = u_i(a^m,t) = \sum_{s_1 \in S_1 \cup \{a_1\}} \cdots \sum_{s_n \in S_n \cup \{a_n\}} a_1^m(s_1)\cdots a_n^m(s_n) u_i(s_1,\ldots,s_n,t).$$

Clearly, u_i^m converges pointwise to u_i.

Step 2. Given any increasing strategy profile ϕ. For each $m \in \mathbb{N}$, $i \in I$, let $\phi_i^m = (1-\frac{1}{m})\phi_i + \frac{1}{m}\sum_{k=1}^{\infty}\frac{1}{2^k}\delta_{s_i^k}$ be a completely mixed strategy profile. Let $F_i = \{\delta_{s_i^k}\}_{k=1}^{\infty}$ be a set of dirac measures on A_i. Then we have $V_i^m(a_i,t_i;\phi_{-i}(\cdot)) = V_i(a_i^m,t_i;\phi_{-i}^m)$, where

$$V_i(a_i^m,t_i;\phi_{-i}^m(\cdot)) = \sum_{s_i \in S_i \cup \{a_i\}} \sum_{j \neq i, p_j \in F_j \cup \{\phi_j\}} a_i^m(s_i)\phi_1^m(p_1)\cdots\phi_n^m(p_n)V_i(s_i,t_i;p_{-i}(\cdot)),$$

and each p_j is an increasing strategy of player j. Notably, (1) if $V_i(a_i,t_i;\phi_{-i}(\cdot))$ is supermodular in a_i, then $V_i^m(a_i,t_i;\phi_{-i}(\cdot))$ is also supermodular in a_i; (2) if $V_i(a_i,t_i;\phi_{-i}(\cdot))$ satisfies IDC in (a_i,t_i), then $V_i^m(a_i,t_i;\phi_{-i}(\cdot))$ also satisfies IDC in (a_i,t_i). By Reny (2011, Theorem 4.1 and Propositon 4.4), we know that G^m possesses a monotone equilibrium g^m. Applying Helly's selection theorem, there exists a subsequence $\{g^{m_k}\}_{m_k=1}^{\infty}$ of $\{g^m\}_{m=1}^{\infty}$ such that $\{g^{m_k}\}_{m_k=1}^{\infty}$ converges to a measurable monotone strategy g for almost all $t \in T$. Consequently, we have $\lim_{k\to\infty}\rho(g_i^{m_k}(t_i,\cdot),g_i(t_i,\cdot)) = 0$, for each player i, for almost all t_i.

Step 3. For each $m \in \mathbb{N}$, let $\overline{g}_i^m = (1-\frac{1}{m})g_i^m + \frac{1}{m}\sum_{k=1}^{\infty}\frac{1}{2^k}\delta_{s_i^k}$. Clearly, $\overline{g}_i^m$ is a sequence of completely mixed strategy profiles. By combining the definition of $\overline{g}_i^m$ with the fact that $\lim_{k\to\infty}\rho(g_i^{m_k}(t_i,\cdot),g_i(t_i,\cdot)) = 0$, we obtain $\lim_{k\to\infty}\rho(\overline{g}_i^{m_k}(t_i,\cdot),g_i(t_i,\cdot)) = 0$, for each player i, for almost all t_i. Note that g^m is an equilibrium of game G^m for each $m \in \mathbb{N}$, meaning that for almost all t_i, $V_i^m(g^m(t_i),t_i;g_{-i}^m) \geq V_i^m(a_i,t_i;g_{-i}^m)$ for all $a_i \in A_i$, which equivalents to $V_i(\overline{g}_i^m(t_i),t_i,\overline{g}_{-i}^m) \geq V_i(a_i^m,t_i,\overline{g}_{-i}^m)$ for all $a_i \in A_i$. By the definitions of $\overline{g}_i^m$ and a_i^m, the above formula implies that $V_i(g_i^m(t_i),t_i,\overline{g}_{-i}^m) \geq V_i(a_i,t_i,\overline{g}_{-i}^m)$ for all $a_i \in A_i$. Thus, we have

$$0 \leq \lim_{k\to\infty}\rho(\overline{g}_i^{m_k}(t_i,\cdot),\mathrm{BR}_i(t_i,\overline{g}_{-i}^{m_k})) \leq \lim_{k\to\infty}\rho(\overline{g}_i^{m_k}(t_i,\cdot),g_i^{m_k}(t_i,\cdot)) = 0.$$

By the inequality $V_i^m(g^m(t_i), t_i; g_{-i}^m) \geq V_i^m(a_i, t_i; g_{-i}^m)$, and letting m tend to infinity, we conclude that g is an equilibrium of game G, and therefore g is a perfect equilibrium. Since g is a monotone strategy, it follows that g is a perfect monotone equilibrium. This completes the proof of Theorem 1.

Bajoori, Flesch and Vermeulen (2016) presented an example of a two-player game with a unique Nash equilibrium in which both players adopt a weakly dominated strategy, making the equilibrium inadmissible. We revisit their example in Example 4, demonstrating the existence of a unique inadmissible perfect equilibrium. Since a perfect equilibrium may not necessarily be admissible, they introduced the concept of "limit undominated" and proved that any perfect equilibrium is limit undominated if for each $t \in T$, the payoff functions $u_i(a, t)$ is continuous in a. Their proof of limit undominated further pointed out that if a probability measure σ_i is a best response for player i against a completely mixed strategy profile at some type t_i, then σ_i is undominated. Corollary 1 follows from the following lemma.

Lemma 1. *A perfect equilibrium is admissible if A_i is finite for each $i \in I$.*

Proof. Suppose an action $a_i \in A_i$ is weakly dominated, meaning that there exists a probability measure $\sigma_i \in \mathcal{M}(A_i)$ such that for λ_i almost all t_i,

(1). $\mathbb{E}_{g_{-i}}(u_i|t_i, a_i) \leq \mathbb{E}_{g_{-i}}(u_i|t_i, \sigma_i)$, for any strategy profile g_{-i};

(2). there exists a strategy profile $\hat{g}_{-i}$ such that $\mathbb{E}_{\hat{g}_{-i}}(u_i|t_i, a_i) < \mathbb{E}_{\hat{g}_{-i}}(u_i|t_i, \sigma_i)$.

Thus, for player i, δ_{a_i} is dominated by σ_i for almost all types t_i.

Suppose h is a perfect equilibrium. Then there exists a sequence of completely mixed strategy profiles $\{h^m\}_{m \in \mathbb{Z}_+}$ such that, for every player i and for almost all t_i, the following properties hold:

(i). $\lim_{m\to\infty} \rho(h_i^m(t_i, \cdot), h_i(t_i, \cdot)) = 0$;

(ii). $\lim_{m\to\infty} \rho(h_i^m(t_i, \cdot), \mathrm{BR}_i(t_i, h_{-i}^m)) = 0$.

From $\lim_{m\to\infty} \rho(h_i^m(t_i, \cdot), \mathrm{BR}_i(t_i, h_{-i}^m)) = 0$, we deduce that for almost all t_i, there exists a sequence of corresponding best response $\{\sigma_i^m\}_{m \in \mathbb{Z}_+}$ ($\sigma_i^m \in \mathrm{BR}_i(t_i, h_{-i}^m)$) such that $\lim_{m\to\infty} \rho(h_i^m(t_i, \cdot), \sigma_i^m) = 0$. Since h_{-i}^m is a completely mixed strategy profile, we know that σ^m is undominated, and hence $\sigma_i^m(a_i) = 0$. By conditions (i) and (ii), we conclude that $\lim_{m\to\infty} \rho(h_i(t_i, \cdot), \sigma_i^m) = 0$. Because A_i is finite, it follows that $h_i(t_i, a_i) = 0$ for almost all t_i. Therefore, h_i assigns zero probability to a weakly dominated action a_i, and hence h is admissible. □

When there are infinitely many actions, a perfect equilibrium may fail to be admissible, even in games with complete information. For an example involving interval action spaces and continuous payoffs that yields an inadmissible Nash equilibrium, see Simon and Stinchcombe (1995). In the following example, we demonstrate the existence of a unique inadmissible perfect equilibrium.

Example 4. *There are two players. The action sets are $A_1 = A_2 = \{\frac{1}{k}\}_{k=1}^{\infty} \cup \{0\}$. The payoff functions u_1 and u_2 are symmetric; that is, $u_1(a, b) = u_2(b, a)$ for all $a, b \in \{\frac{1}{k}\}_{k=1}^{\infty} \cup \{0\}$. The payoff u_1 is given in the table below, where player 1 is the row player and player 2 is the column player.*

u_1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\cdots$	0
1	0	0	0	0	$\cdots$	0
$\frac{1}{2}$	1	-$\frac{1}{8}$	0	0	$\cdots$	0
$\frac{1}{3}$	0	$\frac{1}{2}$	-$\frac{1}{16}$	0	$\cdots$	0
$\frac{1}{4}$	0	0	$\frac{1}{4}$	-$\frac{1}{32}$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
0	0	0	0	0	$\cdots$	0

Table 3: Payoff function u_1

Claim 4. *The strategy profile* $(0, 0)$ *is an inadmissible perfect equilibrium.*

Proof. Notice that the action 0 is weakly dominated by a mixed action $\sigma = (0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots, 0)$, and $(0, 0)$ is the unique Nash equilibrium. We need to show that the strategy profile $(0, 0)$ is perfect.

Since this is a symmetric game, we only need to consider player 1. For each $m \in \mathbb{N}$, let

$$\sigma^m = (1 - \frac{1}{m})\delta_0 + \frac{1}{m}\left[(1 - \frac{1}{m})\delta_{\frac{1}{m}} + \frac{1}{m}\sum_{k=1}^{\infty}\frac{1}{2^k}\delta_{\frac{1}{k}}\right].$$

Clearly, $\{\sigma^m\}_{m=1}^{\infty}$ is a sequence of completely mixed strategies converging to δ_0. That is, $\lim_{m\to\infty} \rho(\sigma^m, \delta_0) = 0$. Given that player 2 plays σ^m, player 1's best response is the action $\frac{1}{m+1}$ when m is sufficiently large. Thus,

$$\lim_{m\to\infty} \rho(\sigma^m, \mathrm{BR}_1(\sigma^m)) = \lim_{m\to\infty} \rho(\sigma^m, \delta_{\frac{1}{m+1}}) = 0.$$

It implies that the strategy profile $(0, 0)$ is perfect. □

8.2 Proof of Proposition 1 and Proposition 3

The analysis of the Bertrand pricing game is analogous to that of the first-price auction. Therefore, we proceed by considering the first-price auction in detail and omit a separate treatment of the Bertrand pricing game here.

We divide the proof into the following steps. In step 1, fix $m \in \mathbb{N}$, we construct a sequence of Bayesian games $\{G^{mk}\}_{k=1}^{\infty}$ that converges to the limit Bayesian game G^m. Here, the game G^m differs from the original game G only in its payoff functions. And each game G^{mk} possesses a monotone equilibrium g^{mk}. In step 2, we demonstrate that there exists a subsequence of $\{g^{mk}\}_{m=1}^{\infty}$ converges to an increasing strategy g^m. In step 3, we show that g^m is a monotone equilibrium in game G^m. Moreover, by applying Helly's selection theorem, we establish the existence of a subsequence of $\{g^m\}_{m=1}^{\infty}$ which converges to a monotone strategy g. In step 4, we show that g is a perfect monotone equilibrium in game G.

Step 1. Fix $m \in \mathbb{Z}_+$. Let $V_i^m(b_i, v_i; \alpha_{-i}(\cdot)) = V_i(b_i^m, v_i; \alpha_{-i}^m(\cdot))$ be bidder i's interim payoff function in game G^m, where $b_i^m = (1-\frac{1}{m})\delta_{\{b_i\}} + \frac{1}{2m}U[\underline{b}_i, \bar{b}_i] + \frac{1}{2m}\delta_{\{Q\}}$, and $\alpha_j^m = (1-\frac{1}{2m})\alpha_j + \frac{1}{2m}U[\underline{b}_i, \bar{b}_i] + \frac{1}{2m}\delta_{\{Q\}}$, for each $j = 1, 2, \cdots, n$. For each $k \in \mathbb{Z}_+$, we construct a finite set $A_i^k \subseteq [\underline{b}_i, \bar{b}_i] \cup \{Q\}$ such that $\cup_{k=1}^{\infty} A_i^k$ is a dense subset of $[\underline{b}_i, \bar{b}_i] \cup \{Q\}$, $A_i^k \subseteq A_i^{k+1}$ and $(\cup_{k=1}^{\infty} A_i^k) \cap (\cup_{k=1}^{\infty} A_j^k) = \{Q\}$ for any $i \neq j$. Let V_i^m be bidder i's interim payoff function and A_i^k be bidder i's action set in game G^{mk}. Let $\beta_{-i}(\cdot)$ be a monotone strategy profile, and $\beta_j^m = (1-\frac{1}{m})\delta_{\{\beta_j\}} + \frac{1}{2m}U[\underline{b}_i, \bar{b}_i] + \frac{1}{2m}\delta_{\{Q\}}$ be a completely mixed strategy profile. Then, by simple algebra, we obtain

$$V_i^m(b_i, v_i; \beta_{-i}(\cdot)) = (1-\frac{1}{m}) \sum_{\gamma_{-i} \in \prod_{j\neq i}\{\beta_j, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}} V_i(b_i, v_i; \gamma_{-j}(\cdot))\mathbb{P}(\gamma_{-i}) + R_i^m(v_i, \beta_{-i}), \tag{1}$$

where $\mathbb{P}(\gamma_j) = 1 - \frac{1}{m}$ if $\gamma_j = \beta_j$, and $\mathbb{P}(\gamma_j) = \frac{1}{2m}$ if $\gamma_j \in \{U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}$. Additionally, we define $\mathbb{P}(\gamma_{-i}) = \prod_{j\neq i} \mathbb{P}(\gamma_j)$, and set

$$R_i^m(v_i, \beta_{-i}) = \frac{1}{2m} \int_{[\underline{b}_i, \bar{b}_i]} \frac{1}{\bar{b}_i - \underline{b}_i} V_i(\tilde{b}_i, v_i, \beta_{-i}^m) \,\mathrm{d}\tilde{b}_i,$$

where $\beta_j^m = (1-\frac{1}{2m})\beta_j + \frac{1}{2m}U[\underline{b}_i, \bar{b}_i] + \frac{1}{2m}\delta_{\{Q\}}$ for all $j \neq i$.

Given a strategy profile $\gamma_{-i} \in \prod_{j\neq i}\{\beta_j, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}$, we divide the players into three subsets. Let $I_1 = \{j : \gamma_j = U[\underline{b}_j, \bar{b}_j]\}$, which consists of bidders employing a uniform distribution strategy over their action sets. Define $I_2 = \{j : \gamma_j = \delta_{\{Q\}}\}$, which includes bidders with a degenerate strategy, placing all probability mass on Q. Let $I_3 = I \backslash (I_1 \cup I_2 \cup \{i\})$, which contains the remaining bidders. Then, by simple algebra, we obtain

$$\begin{aligned}
V_i(b_i, v_i; \gamma_{-i}(\cdot)) &= \int_{[0,1]^{n-1}} u_i(b_i, v_i; \gamma_{-i}(v_{-i})) f(v_{-i}|v_i) \,\mathrm{d}v_{-i} \\
&= \int_{[0,1]^{n-1}} \int_{\prod_{j\in I_1}[\underline{b}_j, \bar{b}_j]} \prod_{j\in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} u_i(b_i, v_i; b_{I_1}, Q_{I_2}, \beta_{I_3}) \bigotimes_{j\in I_1} \mathrm{d}b_j f(v_{-i}|v_i) \,\mathrm{d}v_{-i} \\
&= \prod_{j\in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{\prod_{j\in I_1}[\underline{b}_j, \bar{b}_j]} \int_{[0,1]^{n-1}} u_i(b_i, v_i; b_{I_1}, Q_{I_2}, \beta_{I_3}) f(v_{-i}|v_i) \,\mathrm{d}v_{-i} \bigotimes_{j\in I_1} \mathrm{d}b_j \\
&= \prod_{j\in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{\prod_{j\in I_1}[\underline{b}_j, \bar{b}_j]} V_i(b_i, v_i; \delta_{b_{I_1}}, \delta_{Q_{I_2}}, \beta_{I_3}) \bigotimes_{j\in I_1} \mathrm{d}b_j,
\end{aligned}$$

where $b_{I_1} = (b_j)_{j\in I_1}$, $Q_{I_2} = (Q)_{j\in I_2}$, $\beta_{I_3} = (\beta_j)_{j\in I_3}$, $\delta_{b_{I_1}} = (\delta_{\{b_j\}})_{j\in I_1}$ and $\delta_{Q_{I_2}} = (\delta_{\{Q\}})_{j\in I_2}$. For each bidder $i \in I$, let $p_i^w : \prod_{j\in I}[\underline{b}_j, \bar{b}_j] \to \mathbb{R}$ represent the probability that bidder i wins given a profile of bids. Additionally, we define $p_i^w(Q, b_{-i}) = 0$ for all b_{-i}, meaning that if bidder i quits the auction, her probability of winning is zero regardless of the bids from other players. For any monotone strategy profile ψ, for any $b_i \in [\underline{b}_i, \bar{b}_i]$, we have

$$\begin{aligned}
V_i(b_i, v_i; \psi_{-i}) &= \int_{[0,1]^{n-1}} u_i(b_i, v_i, \psi_{-i}(v_{-i})) f(v_{-i}|v_i) \,\mathrm{d}v_{-i} \\
&= \int_{[0,1]^{n-1}} (v_i - b_i) p_i^w(b_i, \psi_{-i}(v_{-i})) f(v_{-i}|v_i) \,\mathrm{d}v_{-i}
\end{aligned}$$

$$= (v_i - b_i) \int_{[0,1]^{n-1}} p_i^w(b_i, \psi_{-i}(v_{-i})) f(v_{-i}|v_i) \, \mathrm{d}v_{-i}.$$

Define the aggregate winning probability for bidder i against a strategy profile ψ_{-i} as

$$\overline{p}_i^w(b_i, \psi_{-i}) = \int_{[0,1]^{n-1}} p_i^w(b_i, \psi_{-i}(v_{-i})) f(v_{-i}|v_i) \, \mathrm{d}v_{-i}.$$

Therefore, the interim payoff for bidder i given a bid b_i and valuation v_i against the strategy profile ψ_{-i} is

$$V_i(b_i, v_i; \psi_{-i}) = (v_i - b_i)\overline{p}_i^w(b_i, \psi_{-i}).$$

Similarly, define the aggregate winning probability for bidder i against a strategy profile γ_{-i} as

$$\overline{p}_i^w(b_i, \gamma_{-i}) = \prod_{j \in I_1} \frac{1}{\overline{b}_j - \underline{b}_j} \int_{\prod_{j \in I_1}[\underline{b}_j, \overline{b}_j]} \overline{p}_i^w(b_i, \delta_{b_{I_1}}, \delta_{Q_{I_2}}, \beta_{I_3}) \underset{j \in I_1}{\otimes} \mathrm{d}b_j.$$

Therefore, the interim payoff for bidder i given a bid b_i and valuation v_i against the strategy profile γ_{-i} is

$$V_i(b_i, v_i; \gamma_{-i}(\cdot)) = (v_i - b_i)\overline{p}_i^w(b_i, \gamma_{-i}).$$

Let $\overline{p}_i^w(b_i, \beta_{-i}^m) = \sum_{\gamma_{-i} \in \prod_{j \neq i}\{\beta_j, U[\underline{b}_j, \overline{b}_j], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i})\overline{p}_i^w(b_i, \gamma_{-i})$. Then, for any $b_i \in [\underline{b}_i, \overline{b}_i]$, we have

$$V_i^m(b_i, v_i, \beta_{-i}) = (1 - \frac{1}{m})(v_i - b_i)\overline{p}_i^w(b_i, \beta_{-i}^m) + R_i^m(v_i, \beta_{-i}).$$

Next, we are going to show for each bidder $i \in I$, $V_i^m(b_i, v_i; \beta_{-i}(\cdot))$ satisfies IDC(b_i, v_i) for any monotone strategy $\beta_{-i} \in \mathcal{F}_{-i}$. For any $b_i^H, b_i^L > Q \in A_i^k$ and $b_i^H > b_i^L$,

$$\begin{aligned}
&V_i^m(b_i^H, v_i; \beta_{-i}(\cdot)) - V_i^m(b_i^L, v_i; \beta_{-i}(\cdot)) \\
&= (1 - \frac{1}{m})[(v_i - b_i^H)\overline{p}_i^w(b_i^H, \beta_{-i}^m(\cdot)) - (v_i - b_i^L)\overline{p}_i^w(b_i^L, \beta_{-i}^m(\cdot))] \\
&= (1 - \frac{1}{m})[b_i^L\overline{p}_i^w(b_i^L, \beta_{-i}^m(\cdot)) - b_i^H\overline{p}_i^w(b_i^H, \beta_{-i}^m(\cdot)) + v_i(p_i^w(b_i^H, \beta_{-i}^m(\cdot)) - \overline{p}_i^w(b_i^L, \beta_{-i}^m(\cdot)))].
\end{aligned}$$

Since $p_i^w(b_i, b_{-i})$ is increasing in b_i for any b_{-i}, we have $\overline{p}_i^w(b_i^H, \beta_{-i}^m(\cdot)) - \overline{p}_i^w(b_i^L, \beta_{-i}^m(\cdot)) \geq 0$. Thus, $V_i^m(b_i^H, v_i; \beta_{-i}(\cdot)) - V_i^m(b_i^L, v_i; \beta_{-i}(\cdot))$ is increasing in v_i. Note that if $b_i^L = Q$, $V_i^m(b_i^H, v_i; \beta_{-i}(\cdot)) - V_i^m(Q, v_i; \beta_{-i}(\cdot)) = (1 - \frac{1}{m})(v_i - b_i^H)\overline{p}_i^w(b_i^H, \beta_{-i}^m(\cdot))$, which is obviously increasing in v_i. This shows that $V_i^m(b_i, v_i; \beta_{-i}(\cdot))$ satisfies IDC(b_i, v_i) for any $\beta_{-i} \in \mathcal{F}_{-i}$. By applying Theorem 1, we conclude that each game G^{mk} possesses a monotone equilibrium, denoted by g^{mk}.

Step 2. By Helly's selection theorem, there exists a subsequence $\{g^{mn_k}\}_{k=1}^{\infty}$ of $\{g^{mk}\}_{k=1}^{\infty}$ such that $\{g^{mn_k}\}_{k=1}^{\infty}$ converges pointwise to a measurable monotone strategy g^m for almost all $v \in [0,1]^n$. Thus, we have $\lim_{k \to \infty} \rho(g_i^{mn_k}(v_i, \cdot), g_i^m(v_i, \cdot)) = 0$, for each bidder i, for λ_i almost all v_i.

Before we show that g^m is an equilibrium of G^m, we first show that no bidder will bid above her valuation in the strategy g^m. Consider a general scenario where $k \in \mathbb{Z}_+$, $i \in I$, and $v_i \in [0,1]$. Since g^{mk} is an equilibrium of game G^{mk}, we have $V_i^m(g_i^{mk}(v_i), v_i; g_{-i}^{mk}(\cdot)) \geq V_i^m(Q, v_i; g_{-i}^{mk}(\cdot))$. If bidder i has a positive winning probability at her valuation v_i with bidding $g_i^m(v_i)$, that is, $\mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^{mk}(v_j) \leq g_i^{mk}(v_i)\}|v_i) > 0$, then $g_i^{mk}(v_i) \leq v_i$. In other words, she won't bid above her valuation if she has a positive winning probability. Otherwise, bidder i will receive a payoff worse than quitting the game at her valuation v_i, which leads to a contradiction.

Claim 5. *For any $k \in \mathbb{N}$ and $i \in I$, we have $g_i^{mk}(v_i) \leq v_i$.*

Proof. We proceed with this proof by contradiction. Suppose $v_i > 0$ and $g_i^{mk}(v_i) > v_i$. Since g^{mk} is a monotone strategy profile, the set $\{v_{-i} | \max_{j\neq i} g_j^{mk}(v_j) \leq g_i^{mk}(v_i)\}$ is a product of intervals. As g^{mk} is a monotone equilibrium in game G^{mk}, we know that $\mathbb{P}(\{v_{-i} | \max_{j\neq i} g_j^{mk}(v_j) \leq g_i^{mk}(v_i)\} | v_i) = 0$. In other words, there exists a nonempty subset $I_1 \subseteq I$ such that for each bidder $j \in I_1$, $g_j^{mk}(v_j) \geq g_i^{mk}(v_i)$ for all $v_j \in (0, 1]$. Meanwhile, for each bidder $j \in I \backslash I_1$, there exists a nonnegligible subset of $[0, 1]$ such that $g_j^{mk}(v_j) \leq g_i^{mk}(v_i)$. For each $j \in I_1$, since each game G^{mk} has finite action sets, there exists the minimum mass action of g_j^{mk}, denoted by a_j^{mk}. Let $j^* \in \operatorname{argmax}_{j\in I_1} a_j^{mk}$. Then bidder j^*'s payoff from bidding $g_j^{mk}(v_j)$ will be worse than quitting the auction if her valuation $v_j \in (0, \min\{a_{j^*}^{mk}, v_{j^*}^*\})$, where v_j^* is the maximum valuation such that $g_{j^*}^{mk}(v_j^*) = a_{j^*}^{mk}$. Thus, we arrive at a contradiction to the equilibrium property. Consequently, we have $g_i^{mk}(v_i) \leq v_i$, for any $k \in \mathbb{N}$ and $i \in I$. □

By Claim 5 and the fact that $\{g^{mn_k}\}_{k=1}^{\infty}$ converges pointwise to g^m, we conclude that $g_i^m(v_i) \leq v_i$ for almost all $v_i \in [0, 1]$, for all $i \in I$.

Step 3. Throughout the remainder of the proof, it will be convenient to adopt a convention regarding the bid$b_i = Q$. Specifically, notice that Q is an isolated point. Thus, $b_i' \to Q^+$ (which means $b_i' > Q$ and $b_i' \to Q$) will represent $b_i' = Q$.

Let $W_i = \{v_{-i} | b_i \geq \max_{j\neq i} g_j^m(v_j)\}$ be the set of types (a subset of $[0, 1]^{n-1}$) for which bidder i's bid b_i is the maximum bid against $g_{-i}^m(\cdot)$, where $g^m(\cdot)$ is a monotone strategy profile. Define $\mathbb{E}(\cdot|v_i, W_i) = 0$ if $\mathbb{P}(W_i|v_i) = 0$. Notice that when considering the maximum payoff of bidder i at her valuation v_i, her maximum profit will be greater than or equal to the payoff she can achieve by choosing $b_i = v_i$ (or $b_i = Q$)). Moreover, if bidder i chooses $b_i > v_i$ at her valuation v_i, her payoff will be at most equal to the payoff she can receive at $b_i = v_i$ (or $b_i = Q$). Thus, for a fixed v_i, in order to find bidder i's maximum payoff at her valuation v_i, we only need to consider $b_i \leq v_i$.

For $b_i \leq v_i$, we have

$$\begin{aligned} V_i(b_i, v_i; g_{-i}^m(\cdot)) &= \mathbb{P}(W_i|v_i)\mathbb{E}[(v_i - b_i)p_i^w(b_i, g_{-i}^m(v_{-i}))|v_i, W_i] \\ &\leq \mathbb{P}(W_i|v_i)\mathbb{E}[(v_i - b_i)|v_i, W_i]\mathbb{E}[p_i^w(b_i, g_{-i}^m(v_{-i}))|v_i, W_i] \\ &\leq \mathbb{P}(W_i|v_i)\mathbb{E}[(v_i - b_i)|c_i, W_i] \\ &= \lim_{b_i'\to b_i^+} V_i(b_i', v_i; g_{-i}^m(\cdot)), \end{aligned} \tag{2}$$

where the first inequality follows by Milgrom and Weber (1982, Theorem 23), since $1 - p_i^w(b_i, g_{-i}^m(v_{-i}))$ is increasing in v_{-i}. The second inequality follows because $0 \leq p_i^w(b_i, g_{-i}^m(v_{-i})) \leq 1$ and $\mathbb{P}(W_i|c_i)\mathbb{E}[(v_i - b_i)|v_i, W_i] \geq 0$. The last equation holds since (i) $p_i^w(b_i', g_{-i}^m(v_{-i})) = 1$ for all $b_i' > b_i$ and $v_{-i} \in W_i$, and (ii) for all $v_{-i} \notin W_i$, $p_i^w(b_i', g_{-i}^m(v_{-i})) \to 0$ for $b_i' > b_i$ and $b_i' \to b_i$.

Notably, Inequality (2) also holds for any other monotone strategy profile. Since V_i is bounded, by Lebesgue dominated convergence theorem, we have

$$V_i(b_i, v_i; \gamma_{-i}(\cdot)) \le \lim_{b'_i \to b_i^+} V_i(b'_i, v_i; \gamma_{-i}(\cdot)). \tag{3}$$

By combining Equation (1) and Inequality (3), and observing that $R_i^m(v_i, g_{-i}^m)$ is independent of b_i, we conclude that the inequality holds for all $b_i \le v_i$.

$$V_i^m(b_i, v_i; g_{-i}^m(\cdot)) \le \lim_{b'_i \to b_i^+} V_i^m(b'_i, v_i; g_{-i}^m(\cdot)). \tag{4}$$

For each player j, since her strategy $g_j^m(v_j)$ contains at most countably many mass points and A_j^k becomes dense in A_j, it follows that for any $b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}$, any $\epsilon > 0$, and for almost all v_i, there exists $K \in \mathbb{Z}_+$, and $\bar{b}_i \in A_i^K$ such that

$$\begin{aligned}
\lim_{b'_i \to b_i^+} V_i^m(b'_i, v_i; g_{-i}^m(\cdot)) &\le V_i^m(\bar{b}_i, v_i; g_{-i}^m(\cdot)) + \epsilon \\
&\le V_i^m(\bar{b}_i, v_i; g_{-i}^{mn_k}(\cdot)) + 2\epsilon \quad \text{for } k \ge K \\
&\le V_i^m(g_i^{mn_k}(v_i), v_i; g_{-i}^{mn_k}(\cdot)) + 2\epsilon \quad \text{for } k \ge K,
\end{aligned} \tag{5}$$

where the first and second lines hold because $\bar{b}_i$ can be selected such that the probability that any $g_j^m(\cdot)$ equals $\bar{b}_i$ is arbitrarily small. The third line follows since $\bar{b}_i \in A_i^K$ is a feasible action for player i in G^{mn_k} for every $k \ge K$, and g^{mn_k} is an equilibrium in game G^{mn_k}.

The winning probability for bidder i, denoted $p_i^w(b_i, b_{-i})$, is increasing in b_i and decreasing in b_{-i}. Therefore, we conclude that $p_i^w(g_i^{mn_k}(v_i), g_{-i}^{mn_k}(v_{-i}))$ forms a sequence of functions that is monotone in each of its arguments, being increasing in b_i and decreasing in b_{-i}. Note that

$$V_i(g_i^{mn_k}(v_i), v_i; g_{-i}^{mn_k}) = \mathbb{E}[(v_i - g_i^{mn_k}(v_i)) p_i^w(g_i^{mn_k}(v_i), g_{-i}^{mn_k}(v_{-i})) | v_i].$$

Hence, by Helly's selection theorem (extracting a subsequence if necessary), there exists a function $\eta_i \colon [0,1]^n \to [0,1]$ for almost all $v \in [0,1]^n$ such that

$$\mathbb{E}[(v_i - g_i^{mn_k}(v_i)) p_i^w(g_i^{mn_k}(v_i), g_{-i}^{mn_k}(v_{-i})) | v_i]$$

converges to

$$\mathbb{E}[(v_i - g_i^m(v_i)) \eta_i(v) | v_i],$$

by the dominated convergence theorem. Recall that $A_i^{n_k} \cap A_j^{n_k} = \{Q\}$ for all $i \ne j$. Thus, $p_i^w(g_i^{mn_k}(v_i), g_{-i}^{mn_k}(v_{-i})) \in \{0,1\}$ for all $n_k \in \mathbb{Z}_+$ and $v \in [0,1]^n$. One can interpret $\eta_i(\cdot)$ as a tie-breaking rule in the limit, where $\eta_i(v) \in \{0,1\}$ and $\sum_{j=1}^n \eta_j(v) \le 1$.

Given a strategy profile $\gamma^{n_k} \in \prod_{j \in I} \{g_j^{mn_k}, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}$, define the probability of each strategy as follows:

$$\mathbb{P}(\gamma_j^{n_k}) = 1 - \frac{1}{m} \quad \text{if} \quad \gamma_j^{n_k} = g_j^{n_k},$$

and

$$\mathbb{P}(\gamma_j^{n_k}) = \frac{1}{2m} \quad \text{if} \quad \gamma_j^{n_k} = U[\underline{b}_j, \bar{b}_j] \text{ or } \gamma_j^{n_k} = \delta_{\{Q\}}.$$

Similarly, define $\mathbb{P}(\gamma_{-i}^{n_k}) = \prod_{j\neq i}\mathbb{P}(\gamma_j^{n_k})$. Then, by simple algebra, we obtain

$$V_i^m(b_i, v_i; g_{-i}^{mn_k}(\cdot)) = (1-\frac{1}{m}) \sum_{\gamma_{-i}^{n_k}\in\prod\limits_{j\neq i}\{g_j^{mn_k}, U[\underline{b}_j,\bar{b}_j],\delta_{\{Q\}}\}} V_i(b_i, v_i; \gamma_{-j}^{n_k}(\cdot))\mathbb{P}(\gamma_{-i}^{n_k}) + R_i^m(v_i, g_{-i}^{mn_k}). \tag{6}$$

Given a strategy profile $\gamma_{-i}^{n_k} \in \prod_{j\neq i}\{g_j^{mn_k}, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}$, we divide the players into three subsets. Let $I_1^{n_k} = \{j\colon \gamma_j^{n_k} = U[\underline{b}_j, \bar{b}_j]\}$, which consists of bidders employing a uniform distribution strategy over their action sets. Define $I_2^{n_k} = \{j\colon \gamma_j^{n_k} = \delta_{\{Q\}}\}$, which includes bidders with a degenerate strategy, placing all probability mass on Q. Let $I_3^{n_k} = I\backslash(I_1^{n_k}\cup I_2^{n_k}\cup\{i\})$, which contains the remaining bidders. Then, by simple algebra, we obtain

$$\begin{aligned}
V_i(b_i, v_i; \gamma_{-i}^{n_k}(\cdot)) &= \int_{[0,1]^{n-1}} u_i(b_i, v_i; \gamma_{-i}^{n_k}(v_{-i}))f(v_{-i}|v_i)\,\mathrm{d}v_{-i} \\
&= \int_{[0,1]^{n-1}}\int_{\prod\limits_{j\in I_1^{n_k}}[\underline{b}_j,\bar{b}_j]} \prod_{j\in I_1^{n_k}} \frac{1}{\bar{b}_j-\underline{b}_j}(v_i-b_i)p_i^w(b_i, b_{I_1^{n_k}}, Q_{I_2^{n_k}}, g_{I_3^{n_k}}^{mn_k}(v_{I_3^{n_k}})) \\
&\quad \bigotimes_{j\in I_1^{n_k}} \mathrm{d}b_j f(v_{-i}|v_i)\,\mathrm{d}v_{-i} \\
&= (v_i-b_i)\int_{[0,1]^{n-1}}\int_{\prod\limits_{j\in I_1^{n_k}}[\underline{b}_j,\bar{b}_j]} \prod_{j\in I_1^{n_k}} \frac{1}{\bar{b}_j-\underline{b}_j}p_i^w(b_i, b_{I_1^{n_k}}, Q_{I_2^{n_k}}, g_{I_3^{n_k}}^{mn_k}(v_{I_3^{n_k}})) \\
&\quad \bigotimes_{j\in I_1^{n_k}} \mathrm{d}b_j f(v_{-i}|v_i)\,\mathrm{d}v_{-i} \\
&= (v_i-b_i)\int_{[0,1]^{n-1}} \hat{p}_i^w(b_i, \gamma_{-i}^{n_k}(v_{-i}))f(v_{-i}|v_i)\,\mathrm{d}v_{-i},
\end{aligned}$$

where $b_{I_1^{n_k}} = (b_j)_{j\in I_1^{n_k}}$, $Q_{I_2^{n_k}} = (Q)_{j\in I_2^{n_k}}$, $g_{I_3^{n_k}}^{mn_k} = (g_j^{mn_k})_{j\in I_3^{n_k}}$, $\delta_{b_{I_1^{n_k}}} = (\delta_{\{b_j\}})_{j\in I_1^{n_k}}$, $\delta_{Q_{I_2^{n_k}}} = (\delta_{\{Q\}})_{j\in I_2^{n_k}}$, and

$$\hat{p}_i^w(b_i, \gamma_{-i}^{n_k}(v_{-i})) = \int_{\prod\limits_{j\in I_1^{n_k}}[\underline{b}_j,\bar{b}_j]} \prod_{j\in I_1^{n_k}} \frac{1}{\bar{b}_j-\underline{b}_j}p_i^w(b_i, b_{I_1^{n_k}}, Q_{I_2^{n_k}}, g_{I_3^{n_k}}^{mn_k}(v_{I_3^{n_k}})) \bigotimes_{j\in I_1^{n_k}} \mathrm{d}b_j.$$

Let $\gamma_j = g_j^m$ if $\gamma_j^{n_k} = g_j^{mn_k}$ for all $k\in\mathbb{Z}_+$, and $\gamma_j = \gamma_j^{n_k}$ if $\gamma_j^{n_k} = U[\underline{b}_i, \bar{b}_i]$ for all $k\in\mathbb{Z}_+$ or $\gamma_j^{n_k} = \delta_{\{Q\}}$ for all $k\in\mathbb{Z}_+$. Similarly, we can find a function $\eta_i^\gamma\colon [0,1]^n\to[0,1]$ such that for almost all $v\in[0,1]^n$, $\mathbb{E}[(v_i-b_i)\hat{p}_i^w(p_i;\gamma_{-j}^{n_k})|v_i]$ (extracting a subsequence if necessary) converges pointwise to $\mathbb{E}[(v_i-b_i)\eta_i^\gamma(t)|v_i]$ by the dominated convergence theorem. Besides, given v_i, we have $R_i^m(v_i, g_{-i}^{mn_k}) = \frac{1}{2m}\int_{[\underline{b}_i,\bar{b}_i]}\frac{1}{\bar{b}_i-\underline{b}_i}V_i(\tilde{b}_i, v_i, \tilde{g}_{-i}^{mn_k})\,\mathrm{d}\tilde{b}_i$, where $\tilde{g}_j^{mn_k} = (1-\frac{1}{m})g_j^{mn_k} + \frac{1}{2m}U_{[\underline{b}_j,\bar{b}_j]} + \frac{1}{2m}\delta_{\{Q\}}$, for each $j\neq i$. Since g_{-i}^m has at most countably many mass points and $g_{-i}^{mn_k}$ converges pointwise to g_{-i}^m, by Lebesgue dominated convergence theorem, we know it converges to $R_i^m(v_i, g_{-i}^m)$. Since the set $\prod_{j\neq i}\{g_j^m, U[\underline{b}_j,\bar{b}_j], \delta_{\{Q\}}\}$ is finite, and because $\epsilon > 0$ is arbitrarily small, we can find a convergent subsequence such that

$$\begin{aligned}
&\sup_{b_i\in[\underline{b}_j,\bar{b}_j]\cup\{Q\}} V_i^m(b_i,v_i;g_{-i}^m(\cdot))\\
&\leq \liminf_{k\to\infty} V_i^m(g_i^{mn_k}(v_i),v_i;g_{-i}^{mn_k}(\cdot))\\
&\leq \lim_{l\to\infty} V_i^m(g_i^{mn_l}(v_i),v_i;g_{-i}^{mn_l}(\cdot))\\
&=(1-\frac{1}{m})\sum_{\gamma_{-i}\in\prod_{j\neq i}\{g_j^m,U[\underline{b}_j,\bar{b}_j],\delta_{\{Q\}}\}}\mathbb{P}(\gamma_{-i})\mathbb{E}[(v_i-g_i^m(v_i))\eta_i^\gamma(v)|v_i]+R_i^m(v_i,g_{-i}^m),
\end{aligned}\tag{7}$$

where η_i^γ is the limit corresponding to the sequence $\{\hat{p}_i^w(g_i^{mn_l}(v_i),\gamma_{-i}^{n_l})\}_{l=1}^\infty$.The first inequality follows by Inequalities (2) and (5), and the first equation follows by Equation (6) with k tends to infinity.

Consider any v_i such that $g_i^m(v_i)>Q$. Define the set $\hat{W}_i=\{v_{-i}\mid g_i^m(v_i)\geq \max_{j\neq i}g_j^m(v_j)\}$ as the set of type profiles v_{-i} for which bidder i's bid $g_i^m(v_i)$ is among the highest bids relative to the bids of the other bidders, as specified by the strategy profile $g_{-i}^m(\cdot)$. We define $\mathbb{E}(\cdot\mid v_i,\hat{W}_i)=0$ if $\mathbb{P}(\hat{W}_i\mid v_i)=0$. When $\mathbb{P}(\hat{W}_i\mid v_i)>0$, we compute the expectation conditional on this event. Then we have

$$\begin{aligned}
0\leq&\mathbb{E}[(v_i-g_i^m(v_i))\eta_i(v)|v_i]\\
&=\mathbb{P}(\hat{W}_i|v_i)\mathbb{E}[(v_i-g_i^m(v_i))\eta_i(v)|v_i,\hat{W}_i]\\
&\leq \mathbb{P}(\hat{W}_i|v_i)\mathbb{E}[v_i-g_i^m(v_i)|v_i,\hat{W}_i]\mathbb{E}[\eta_i(v)|v_i,\hat{W}_i]\\
&\leq \mathbb{P}(\hat{W}_i|v_i)\mathbb{E}[v_i-g_i^m(v_i)|v_i,\hat{W}_i]\\
&=\lim_{\epsilon\to0^+}V_i(g_i^m(v_i)+\epsilon,v_i;g_{-i}^m(\cdot)).
\end{aligned}\tag{8}$$

The first inequality is due to the fact that $v_i-g_i^m(v_i)\geq0$. The second inequality follows by Milgrom and Weber (1982, Theorem 23). The third inequality holds because $0\leq\mathbb{E}[\eta_i(v)\mid v_i,\hat{W}_i]\leq1$ and $\mathbb{P}(\hat{W}_i\mid v_i)\,\mathbb{E}[v_i-g_i^m(v_i)\mid v_i,\hat{W}_i]\geq0$. The final equality results from two observations: (i) for all $\epsilon>0$ and $v_{-i}\in\hat{W}_i$, we have $p_i^w(g_i^m(v_i)+\epsilon,g_{-i}^m(v_{-i}))=1$; and (ii) for all $v_{-i}\notin\hat{W}_i$, as $\epsilon\to0$, $p_i^w(g_i^m(v_i)+\epsilon,g_{-i}^m(v_{-i}))\to0$.

Given a strategy profile $\gamma_{-i}\in\prod_{j\neq i}\{g_j^m,\delta_{\{Q\}},U[\underline{b}_j,\bar{b}_j]\}$. Let $I_1=\{j\colon\gamma_j=U[\underline{b}_j,\bar{b}_j]\}$, which consists of bidders employing a uniform distribution strategy over their action sets. Define $I_2=\{j\colon\gamma_j=\delta_{\{Q\}}\}$, which includes bidders with a degenerate strategy, placing all probability mass on Q. Let $I_3=I\backslash(I_1\cup I_2\cup\{i\})$, which contains the remaining bidders. For all n_k, we define $\gamma_j^{n_k}=g_j^{mn_k}$ if $\gamma_j=g_j^m$, and $\gamma_j^{n_k}=\gamma_j$ otherwise. Let $\hat{W}_i^\gamma=\{v_{-i}|g_i^m(v_i)\geq\max_{j\in I_3}g_j^m(v_j)\}$ be the set of types such that bidder i might win the auction. We define $\mathbb{E}(\cdot|v_i,\hat{W}_i^\gamma)=0$ if $\mathbb{P}(\hat{W}_i^\gamma|v_i)=0$. Then, for $\mathbb{P}(\hat{W}_i^\gamma|v_i)>0$, we have

$$\begin{aligned}
0\leq&\mathbb{E}[(v_i-g_i^m(v_i))\eta_i^\gamma(v)|v_i]\\
&=\mathbb{P}(\hat{W}_i^\gamma|v_i)\mathbb{E}[(v_i-g_i^m(v_i))\eta_i^\gamma(v)|v_i,\hat{W}_i^\gamma]\\
&\leq \mathbb{P}(\hat{W}_i^\gamma|v_i)\mathbb{E}[v_i-g_i^m(v_i)|v_i,\hat{W}_i^\gamma]\mathbb{E}[\eta_i^\gamma(v)|v_i,\hat{W}_i^\gamma]\\
&\leq \mathbb{P}(\hat{W}_i^\gamma|v_i)\mathbb{E}[v_i-g_i^m(v_i)|v_i,\hat{W}_i^\gamma]\zeta^\gamma(v_i)\\
&=\lim_{\epsilon\to0^+}V_i(g_i^m(v_i)+\epsilon,v_i;\gamma_{-i}(\cdot)),
\end{aligned}\tag{9}$$

where $\zeta^{\gamma}(v_i) = \prod_{j \in I_1} \max\{0, \min\{1, \frac{g_i^m(v_i) - \underline{b}_j}{\bar{b}_j - \underline{b}_j}\}\}$, and the second inequality follows by Milgrom and Weber (1982, Thoerem 23). Hence, by Lebesgue dominated convergence theorem, we can obtain that

$$\begin{aligned}
&\lim_{l \to \infty} V_i^m(g_i^{mn_l}(v_i), v_i; g_{-i}^{mn_l}(\cdot)) \\
&\leq \lim_{\epsilon \to 0^+} V_i^m(g_i^m(v_i) + \epsilon, v_i; g_{-i}^m(\cdot)) \\
&\leq \sup_{b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}} V_i^m(b_i, v_i; g_{-i}^m(\cdot)),
\end{aligned} \tag{10}$$

where the first inequality follows by Inequalities (7), (8) and (9).

Lastly, we show that the probability of the set of types in which at least two players submit the highest bid above Q is zero. Combining Inequalities (7) and (10), we conclude that all inequalities in Inequalities (7) – (10) must hold as equations. In particular, if $\mathbb{P}(\hat{W}_i \mid v_i) > 0$, then we have

$$0 \leq \mathbb{E}[v_i - g_i^m(v_i)|v_i, \hat{W}_i]\mathbb{E}[\eta_i(v)|v_i, \hat{W}_i] = \mathbb{E}[v_i - g_i^m(v_i)|v_i, \hat{W}_i]. \tag{11}$$

We know that $v_i - g_i^m(v_i) \geq 0$ by Claim 5. Next, we are going to show that $v_i - g_i^m(v_i) > 0$.

Claim 6. *For almost all v_i such that $\mathbb{P}(\hat{W}_i \mid v_i) > 0$, we have $v_i - g_i^m(v_i) > 0$.*

Proof. This statement is clearly true if $g_i^m(v_i) \leq 0$. We now consider the case where $g_i^m(v_i) > 0$. By the definition of a monotone strategy, we know that bidder i satisfies $g_i^m(\tilde{v}_i) \leq g_i^m(v_i)$ for all $\tilde{v}_i \in [0, v_i]$. Since $\mathbb{P}(\hat{W}_i \mid v_i) > 0$, there exists $\hat{v}_j > 0$ such that $g_j^m(\tilde{v}_j) \leq g_i^m(v_i)$ for all $\tilde{v}_j \in [0, \hat{v}_j]$ and for each bidder $j \neq i$.

Suppose there exists $j' \neq i$ such that $g_{j'}^m(\tilde{v}_{j'}) = g_i^m(v_i) > 0$ for almost all $\tilde{v}_{j'} \in [0, \hat{v}_{j'}]$. Since $g_{j'}^m(v_{j'}) \leq v_{j'}$ for almost all $v_{j'} \in [0, 1]$, it is impossible for bidder j' to satisfy $g_{j'}^m(\tilde{v}_{j'}) = g_i^m(v_i) > 0$ for $\tilde{v}_{j'} \in [0, \min\{\hat{v}_{j'}, g_i^m(v_i)\}]$. Thus, for each $j \neq i$, we have $g_j^m(\tilde{v}_j) < g_i^m(v_i)$ on a nonnegligible subset of $[0, \hat{v}_j]$. Therefore, for each bidder $j \neq i$, there exists $b_j^* < g_i^m(v_i)$ and $0 < \dot{v}_j < \hat{v}_j$ such that $g_j^m(\tilde{v}_j) < b_j^*$ for all $\tilde{v}_j \in [0, \dot{v}_j]$. Let $\dot{b} = \max_{j \neq i} b_j^* < g_i^m(v_i)$. Next, we classify $g_i^m(v_i)$ into the following two cases.

Case 1: Let $g_i^m(v_i)$ be a mass point of bidder i with the strategy profile g_i^m. It means that there exists $\epsilon^* > 0$ such that (1) $g_i^m(\tilde{v}_i) \equiv g_i^m(v_i)$ for all $\tilde{v}_i \in [v_i - \epsilon^*, v_i]$; or (2) $g_i^m(\tilde{v}_i) < g_i^m(v_i)$ for all $\tilde{v}_i < v_i$ and $g_i^m(\tilde{v}_i) \equiv g_i^m(v_i)$ for all $\tilde{v}_i \in [v_i, v_i + \epsilon^*]$. Since for each $\tilde{v}_i \in [v_i - \epsilon^*, v_i + \epsilon^*]$, we have $\tilde{v}_i \geq g_i^m(\tilde{v}_i)$. Thus, if $g_i^m(\tilde{v}_i) \equiv g_i^m(v_i)$ for all $\tilde{v}_i \in [v_i - \epsilon^*, v_i]$, then we have $v_i > v_i - \epsilon^* \geq g_i^m(v_i - \epsilon^*) = g_i^m(v_i)$. In other words, suppose $g_i^m(v_i)$ is a mass point of $g_i^m(\cdot)$, except that $v_i = \text{argmin}_{\tilde{v}_i \in [0,1]}\{\tilde{v}_i | g_i^m(\tilde{v}_i) = g_i^m(v_i)\}$ (that is, $g_i^m(\tilde{v}_i) < g_i^m(v_i)$ for all $\tilde{v}_i < v_i$), we should have $v_i > g_i^m(v_i)$. Note that g_i^m has at most countably many mass points, and hence there exist at most countably many valuations v_i such that $g_i^m(v_i)$ is a mass point and $v_i = \text{argmin}_{\tilde{v}_i \in [0,1]}\{\tilde{v}_i | g_i^m(\tilde{v}_i) = g_i^m(v_i)\}$. Since λ_i is atomless, we know that any countable set has measure zero.

Case 2: Suppose $g_i^m(v_i)$ is not a mass point of bidder i, which implies that $g^m(\cdot)$ is continuous at v_i. By the continuity property, there exists $v_i' < v_i$ such that $g_i^m(v_i') \in [\dot{b}, g_i^m(v_i))$ and $g_i^m(v_i')$ is not a mass point of $g^m(\cdot)$. This is achievable due to the continuity property and the fact that g^m has at most countably many mass points.

Let $\hat{W}_i' = \{v_{-i} | \max_{j \neq i} g_j^m(v_j) \leq g_i^m(v_i')\}$. Then, we have $\mathbb{P}(\hat{W}_i' | v_i') > 0$. We can find a sequence of bids $\{b_i^{n_l}\}_{l=1}^{\infty}$ such that $b_i^{n_l}$ is the greatest bid in the set $A_i^{n_l}$ with $b_i^{n_l} \leq g_i^m(v_i')$.

Since $\cup_{k=1}^{\infty} A_i^k$ is dense in $[\underline{b}_i, \overline{b}_i] \cup \{Q\}$, we deduce that $b_i^{n_l} \to g_i^m(v_i')$. Since $g_i^m(v_i')$ is not a mass point of $g^m(\cdot)$, we have

$$\begin{aligned}
&V_i(b_i^{n_l}, v_i, g_{-i}^{mn_l}(\cdot)) \\
&= \mathbb{E}[(v_i - b_i^{n_l}) p_i^w(b_i^{n_l}, g_{-i}^{mn_l}(\cdot))|v_i, \{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) \leq b_i^{n_l}\}] \mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) \leq b_i^{n_l}\}|v_i) \\
&= \mathbb{E}[v_i - b_i^{n_l}|v_i, \{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) < b_i^{n_l}\}] \mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) < b_i^{n_l}\}|v_i) \\
&\to \mathbb{E}[v_i - g_i^m(v_i')|v_i, \{v_{-i}| \max_{j \neq i} g_j^m(v_j) < g_i^m(v_i')\}] \mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^m(v_j) \leq g_i^m(v_i')\}|v_i)
\end{aligned}$$

where the second equation holds since $A_j^{n_l} \cap A_i^{n_l} = \{Q\}$ for $i \neq j$. Note that $b_i^{n_l}$ converges to $g_i^m(v_i')$ (resp. $g_{-i}^m(\cdot)$), and $g_{-i}^{mn_l}(\cdot)$ converges pointwise to $g_{-i}^m(\cdot)$ for almost all v_{-i}. For simplicity, we consider $g_{-i}^{mn_l}(\cdot)$ converges pointwise to $g_{-i}^m(\cdot)$ for all v_{-i}. Then we have

$$\{v_{-i}| \max_{j \neq i} g_j^m(v_j) < g_i^m(v_i')\} \subseteq \liminf_{l \to \infty} \{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) \leq b_i^{n_l}\},$$

$$\limsup_{l \to \infty} \{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) \leq b_i^{n_l}\} \subseteq \{v_{-i}| \max_{j \neq i} g_j^m(v_j) \leq g_i^m(v_i')\}.$$

Since $g_i^m(v_i')$ is not a mass point of g^m, we have

$$\begin{aligned}
&\mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^m(v_j) \leq g_i^m(v_i')\}|v_i) \\
&= \mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^m(v_j) < g_i^m(v_i')\}|v_i) \\
&= \lim_{l \to \infty} \mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^m(v_j) < b_i^{n_l}\}|v_i).
\end{aligned}$$

Recall that $\{n_l\}_{l=1}^{\infty}$ is a subsequence of $\{n_k\}_{k=1}^{\infty}$. Since g^{mn_l} is a monotone equilibrium of G^{mn_l}, we have

$$\begin{aligned}
&V_i(g_i^{mn_l}(v_i), g_{-i}^{mn_l}(\cdot)) \\
&= \mathbb{E}[(v_i - g_i^{mn_l}(v_i)) p_i^w(g^{mn_l}(\cdot))|v_i, \{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) \leq g_i^{mn_l}(v_i)\}] \\
&\quad \cdot \mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) \leq g_i^{mn_l}(v_i)\}|v_i) \\
&\geq \mathbb{E}[(v_i - b_i^{n_l}) p_i^w(b_i^{n_l}, g_{-i}^{mn_l}(\cdot))|v_i, \{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) \leq b_i^{n_l}\}] \mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^{mn_l}(v_j) \leq b_i^{n_l}\}|v_i) \\
&= V_i(b_i^{n_l}, g_{-i}^{mn_l}(\cdot)).
\end{aligned}$$

The set of v_i such that $g_i^m(v_i)$ is a mass point of g_{-i}^m but not a mass point of g_i^m consists of at most countably many points. Consequently, the measure of this set is zero, allowing us to disregard it. Let l tends to ∞, we consider v_i such that $g_i^m(v_i)$ is not a mass point of g^m, then we have $\mathbb{E}[v_i - g_i^m(v_i)|v_i, \hat{W}_i] \mathbb{P}(\hat{W}_i|v_i) \geq \mathbb{E}[v_i - g_i^m(v_i')|v_i, \{v_{-i}| \max_{j \neq i} g_j^m(v_j) < g_i^m(v_i')\}] \mathbb{P}(\{v_{-i}| \max_{j \neq i} g_j^m(v_j) \leq g_i^m(v_i')\}|v_i)$. Since $v_i' - g_i^m(v_i') \geq 0$, we have $v_i - g_i^m(v_i') > 0$. And hence, $v_i - g_i^m(v_i) > 0$.

□

In summary, we observe that the inequality in Inequality (11) holds strictly. Therefore, we have $\mathbb{E}[\eta_i(v)|v_i, \hat{W}_i] = 1$ for almost all v_i such that $\mathbb{P}(\hat{W}_i|v_i) > 0$. Consequently, given a nonempty subset $B \subseteq \{1, 2, \ldots, n\}$ and letting $T_B = \{v\colon g_i^m(v_i) = \max_j g_j^m(v_j) > Q, \forall i \in$

$B\}$, if $\mathbb{P}(T_B) > 0$, then for every $i \in B$, we have $\eta_i(v) = 1$ for almost all $v \in T_B$. However, since $\sum_{i=1}^n \eta_i(v) \leq 1$ for almost all $v \in [0,1]^n$, it implies that $\#|B| = 1$. Therefore, the probability that under g^m, two or more bidders simultaneously submit the highest bid above Q is zero. Thus, for every i and almost all v_i, $V_i^m(\cdot, v_i; \cdot)$ is continuous at $(g_i^m(v_i), g_{-i}^m(v_{-i}))$. Consequently, $\lim_{l\to\infty} V_i^m(g_i^{mn_l}(v_i), v_i; g_{-i}^{mn_l}(\cdot)) = V_i^m(g_i^m(v_i), v_i; g_{-i}^m(\cdot))$ for almost all v_i. This implies that $V_i^m(g_i^m(v_i), v_i; g_{-i}^m(\cdot)) = \sup_{b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}} V_i^m(b_i, v_i; g_{-i}^m(\cdot))$ for almost all v_i. Therefore, g^m is a monotone equilibrium.

Step 4. By Helly's selection theorem, there exists a subsequence $\{g^{m_k}\}_{k=1}^\infty$ of $\{g^m\}_{m=1}^\infty$ that converges to g for almost all v. Since $g_i^m(v_i) \leq v_i$ for all $v_i \in [0,1]$, all $i \in I$, and all $m \in \mathbb{N}$, it follows that $g_i(v_i) \leq v_i$ for almost all $v_i \in [0,1]$ and for all $i \in I$. In this step, we will demonstrate that g is a perfect monotone equilibrium.

By the limit property, we have $\lim_{k\to\infty} \rho(g_i^{m_k}(v_i), g_i(v_i)) = 0$ for all i and for almost all v_i. Let $\overline{g}_j^{m_k} = (1 - \frac{1}{m_k}) g_j^{m_k} + \frac{1}{2m_k} U[\underline{b}_j, \bar{b}_j] + \frac{1}{2m_k}\delta_{\{Q\}}$, for all j. Since g^{m_k} is an equilibrium in G^{m_k}, we know that $V_i^{m_k}(b_i, v_i; g_{-i}^{m_k}(\cdot)) \leq V_i^{m_k}(g_i^{m_k}(v_i), v_i; g_{-i}^{m_k}(\cdot))$ for all $b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}$, which equivalents to $V_i(b_i, v_i; \overline{g}_{-i}^{m_k}(\cdot)) \leq V_i(g_i^{m_k}(v_i), v_i; \overline{g}_{-i}^{m_k}(\cdot))$ for all $b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}$ and all i. Thus, $g_i^{m_k}(v_i) \in \mathrm{BR}_i(v_i, \overline{g}_{-i}^{m_k}(v_i))$ for almost all v_i. And hence, the completely mixed strategy $\tilde{g}^{m_k}$ in G satisfies $\lim_{m_k\to\infty} \rho(\overline{g}_i^{m_k}(v_i), \mathrm{BR}_i(v_i, \overline{g}_{-i}^{m_k}(v_i))) \leq \lim_{k\to\infty} \rho(\overline{g}_i^{m_k}(t_i), g_i^{m_k}(t_i)) = 0$. To show g is a perfect monotone equilibrium, it remains to show that g is an equilibrium. Thus, we only need to show that the probability of two or more players simultaneously submitting the highest bid above Q under g is 0. This can be derived using similar arguments as in Step 3. We outline the main idea below.

Notice that we can obtain the following inequality (modified from Inequality (5)):

$$
\begin{aligned}
\lim_{b_i' \to b_i^+} V_i(b_i', v_i; g_{-i}(\cdot)) &\leq V_i(\bar{b}_i, v_i; g_{-i}(\cdot)) + \epsilon \\
&\leq V_i(\bar{b}_i, v_i; g_{-i}^{m_k}(\cdot)) + 2\epsilon \quad \text{for } k \geq K \\
&\leq V_i(g_i^{m_k}(v_i), v_i; g_{-i}^{m_k}(\cdot)) + 3\epsilon \quad \text{for } k \geq K,
\end{aligned}
$$

where the first and second inequalities of the following align holds for some $\bar{b}_i$ that sufficiently close to b_i, and $\bar{b}_i$ is not a mass of g and g^m for all m. The last inequality holds because g^{m_k} is an equilibrium of G^{m_k}, meaning that $V_i^{m_k}(\bar{b}_i, v_i; g_{-i}^{m_k}(\cdot)) \leq V_i^{m_k}(g_i^{m_k}(v_i), v_i; g_{-i}^{m_k}(\cdot))$. When k is sufficiently large, we obtain $V_i(\bar{b}_i, v_i; g_{-i}^{m_k}(\cdot)) \leq V_i(g_i^{m_k}(v_i), v_i; g_{-i}^{m_k}(\cdot)) + \epsilon$. Thus, combining the above two inequalities, we have

$$
\sup_{b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}} V_i(b_i, v_i; g(\cdot)) \leq \liminf_{k\to\infty} V_i(g_i^{m_k}(v_i), v_i; g_{-i}^{m_k}(\cdot)).
$$

Notice that $V_i(g_i^{m_k}(v_i), v_i; g_{-i}^{m_k}(\cdot)) = \mathbb{E}[(v_i - g_i^{m_k}(v_i)) p_i^w(g^{m_k}) \mid v_i]\, \mathbb{P}(\{v_{-i} \mid \max_{j\neq i} g_j^{m_k}(v_j) \leq g_i^{m_k}(v_i)\} \mid v_i)$. By the monotonicity property and Helly's selection theorem, we obtain a subsequence $\{m_l\}_{l=1}^\infty$ of $\{m_k\}_{k=1}^\infty$ such that

$$
\mathbb{E}[(v_i - g_i^{m_l}(v_i)) p_i^w(g^{m_l}) \mid v_i] \to \mathbb{E}[(v_i - g_i(v_i)) \eta_i^* \mid v_i]
$$

by the dominated convergence theorem, where $\eta_i^* \colon [0,1]^n \to [0,1]$.

By the same arguments as the proof of Inequality (8), we can show that

$$
\lim_{l\to\infty} V_i(g_i^{m_l}(v_i), v_i; g_{-i}^{m_l}) \leq \sup_{b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}} V_i(b_i, v_i; g(\cdot)).
$$

Last but not least, we repeat the proof of Claim 6. Consequently, we conclude that the probability of two or more players simultaneously submitting the highest bid above Q under g is zero. Then, we have $\lim_{l\to\infty} V_i(g_i^{m_l}(v_i), v_i; g_{-i}^{m_l}) = V_i(g_i(v_i), v_i; g_{-i})$. Combined with $\lim_{l\to\infty} V_i(g_i^{m_l}(v_i), v_i; g_{-i}^{m_l}) = \sup_{b_i\in[\underline{b}_i,\bar{b}_i]\cup\{Q\}} V_i(b_i, v_i; g(\cdot))$, we conclude that g is a monotone equilibrium, which completes our proof.

8.3 Proof of Proposition 2

The approach used in this proof is analogous to the method employed in the proof of Theorem 1. For simplicity, we assume that the winning payoff function $w_i(a,t)$ is uniformly bounded by a constant M for all players $i = 1, 2, \ldots, n$. This implies that $|w_i(a,t)| \le M, \quad \forall i \in \{1, 2, \ldots, n\}, a \in A, t \in T$.

Step 1. In this step, we construct a sequence of Bayesian games $\{G^{mk}\}_{k=1}^{\infty}$ that converges to the limit Bayesian game G^m. Each game G^{mk} possesses a monotone equilibrium g^{mk}. Note that the game G^{mk} and G^m are constructed in the same manner as in Step 1 of the proof of Theorem 1. We shall demonstrate that bidder i's interim payoff function V_i^m satisfies IDC(v_i, b_i) for any monotone strategy $\beta_{-i} \in \mathcal{F}_{-i}$. By applying Theorem 1, it follows that each game G^{mk} possesses a monotone equilibrium.

Next, we are going to show for each bidder $i \in I$, $V_i^m(b_i, t_i; \beta_{-i}(\cdot))$ satisfies IDC(b_i, t_i) for any monotone strategy $\beta_{-i} \in \mathcal{F}_{-i}$. For any $b_i^H, b_i^L > Q \in A_i^k$ and $b_i^H > b_i^L$,

$$
\begin{aligned}
&V_i^m(b_i^H, t_i; \beta_{-i}(\cdot)) - V_i^m(b_i^L, t_i; \beta_{-i}(\cdot)) \\
&= (1 - \frac{1}{m})[V_i(b_i^H, t_i; \beta_{-i}^m(\cdot)) - V_i(b_i^L, t_i; \beta_{-i}^m(\cdot))] \\
&= (1 - \frac{1}{m}) \sum_{\gamma_{-i}\in\prod_{j\neq i}\{\beta_j, U[\underline{b}_j,\bar{b}_j], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i})[V_i(b_i^H, t_i; \gamma_{-i}(\cdot)) - V_i(b_i^L, t_i; \gamma_{-i}(\cdot))],
\end{aligned}
$$

where $\mathbb{P}(\gamma_{-i}) = \prod_{j\neq i} p_j(\gamma_j)$ and $\mathbb{P}(\gamma_j) = (1 - \frac{1}{m})\delta_{\beta_j}(\gamma_j) + \frac{1}{2m}\delta_{U[\underline{b}_j,\bar{b}_j]}(\gamma_j) + \frac{1}{2m}\delta_{\delta_{\{Q\}}}(\gamma_j)$. Given a strategy profile $\gamma_{-i} \in \prod_{j\neq i}\{\beta_j, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}$, we divide the players into three subsets. Let $I_1 = \{j\colon \gamma_j = U[\underline{b}_j, \bar{b}_j]\}$, which consists of bidders employing a uniform distribution strategy over their action sets. Define $I_2 = \{j\colon \gamma_j = \delta_{\{Q\}}\}$, which includes bidders with a degenerate strategy, placing all probability mass on Q. Let $I_3 = I\backslash(I_1 \cup I_2 \cup \{i\})$, which contains the remaining bidders. Then, by simple algebra, we obtain

$$
\begin{aligned}
&V_i(b_i^H, t_i; \gamma_{-i}(\cdot)) - V_i(b_i^L, t_i; \gamma_{-i}(\cdot)) \\
&= \int_{T_{-i}} [u_i(b_i^H, t_i; \gamma_{-i}(t_{-i})) - u_i(b_i^L, t_i; \gamma_{-i}(t_{-i}))] f(t_{-i}|t_i)\,\mathrm{d}t_{-i} \\
&= \prod_{j\in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{T_{-i}} \int_{\prod_{j\in I_1}[\underline{b}_j,\bar{b}_j]} [u_i(b_i^H, t_i; b_{I_1}, Q_{I_2}, \beta_{I_3}) - u_i(b_i^L, t_i; b_{I_1}, Q_{I_2}, \beta_{I_3})] \bigotimes_{j\in I_1} \mathrm{d}b_j f(t_{-i}|t_i)\,\mathrm{d}t_{-i} \\
&= \prod_{j\in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{\prod_{j\in I_1}[\underline{b}_j,\bar{b}_j]} \int_{T_{-i}} [u_i(b_i^H, t_i; b_{I_1}, Q_{I_2}, \beta_{I_3}) - u_i(b_i^L, t_i; b_{I_1}, Q_{I_2}, \beta_{I_3})] f(t_{-i}|t_i)\,\mathrm{d}t_{-i} \bigotimes_{j\in I_1} \mathrm{d}b_j \\
&= \prod_{j\in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{\prod_{j\in I_1}[\underline{b}_j,\bar{b}_j]} [V_i(b_i^H, t_i; \delta_{b_{I_1}}, \delta_{Q_{I_2}}, \beta_{I_3}) - V_i(b_i^L, t_i; \delta_{b_{I_1}}, \delta_{Q_{I_2}}, \beta_{I_3})] \bigotimes_{j\in I_1} \mathrm{d}b_j,
\end{aligned}
$$

where $b_{I_1} = (b_j)_{j\in I_1}$, $Q_{I_2} = (Q,\ldots,Q)$ with $\dim(Q_{I_2}) = \#|I_2|$, $\beta_{I_3} = (\beta_j)_{j\in I_3}$, $\delta_{b_{I_1}} = (\delta_{\{b_j\}})_{j\in I_1}$, and $\delta_{Q_{I_2}} = (\delta_{\{Q\}})_{j\in I_2}$. Let $p_i^w(b_i, b_{-i})$ represent the winning probability for bidder i when choosing action b_i, while other bidders choose actions b_{-i}, and define $p_i^w(Q, b_{-i}) = 0$ for all $b_{-i} \in T_{-i}$. By straightforward algebraic manipulation, for any monotone strategy $\psi(\cdot)$, we have

$$\begin{aligned}
&V_i(b_i^H, t_i; \psi_{-i}(\cdot)) - V_i(b_i^L, t_i; \psi_{-i}(\cdot)) \\
&= \int_{T_{-i}} w_i(b_i^H, t_i, t_{-i}) p_i^w(b_i^H, \psi_{-i}(t_{-i})) - w_i(b_i^L, t_i, t_{-i}) p_i^w(b_i^L, \psi_{-i}(t_{-i})) f(t_{-i}|t_i)\, \mathrm{d}t_{-i} \\
&\quad - (b_i^H - b_i^L) \\
&= \int_{T_{-i}} w_i(b_i^H, t_i, t_{-i})(p_i^w(b_i^H, \psi_{-i}(t_{-i})) - p_i^w(b_i^L, \psi_{-i}(t_{-i}))) f(t_{-i}|t_i)\, \mathrm{d}t_{-i} \\
&\quad + \int_{T_{-i}} (w_i(b_i^H, t_i, t_{-i}) - w_i(b_i^L, t_i, t_{-i})) p_i^w(b_i^L, \psi_{-i}(t_{-i})) f(t_{-i}|t_i)\, \mathrm{d}t_{-i} \\
&\quad - (b_i^H - b_i^L).
\end{aligned}$$

Since $w_i(b_i^H, t_i, t_{-i}) - w_i(b_i^L, t_i, t_{-i})$ is increasing in t_i and $w_i(b_i, t_i, t_{-i})$ is strictly increasing in t_i, it follows that for each bidder $i \in I$, the interim payoff function $V_i^m(b_i, t_i; \beta_{-i}(\cdot))$ satisfies IDC(b_i, t_i) for all $\beta_{-i} \in \mathcal{F}_{-i}$. Furthermore, if $b_i^L = Q$, then $V_i^m(b_i^H, t_i; \gamma_{-i}(\cdot)) - V_i^m(Q, t_i; \gamma_{-i}(\cdot)) = \left(1 - \frac{1}{m}\right) \sum_{\gamma_{-i} \in \prod_{j\neq i} \{\beta_j, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i}) V_i(b_i^H, t_i; \gamma_{-i}(\cdot))$ is increasing in t_i. By Theorem 1, there exists a monotone equilibrium in each game G^{mk}, denoted by g^{mk}.

Step 2. By Helly's selection theorem, there exists a subsequence $\{g^{mn_k}\}_{k=1}^\infty$ of $\{g^{mk}\}_{k=1}^\infty$ such that $\{g^{mn_k}\}_{k=1}^\infty$ converges pointwise to a measurable monotone strategy g^m for almost all $v \in [0,1]^n$. Thus, we have $\lim_{k\to\infty} \rho(g_i^{mn_k}(v_i, \cdot), g_i^m(v_i, \cdot)) = 0$, for each bidder i, for λ_i almost all v_i.

Step 3. In this step, we demonstrate that for each $m \in \mathbb{N}$, g^m is a monotone equilibrium in G^m, and that there exists a subsequence of $\{g^m\}_{m=1}^\infty$ which converges to a monotone strategy g. Throughout the remainder of the proof, we assume that (i) $u_i(Q, t) = 0$ for all i and t; and (ii) Q is an isolated point, so that $b_i' \to Q$ implies $b_i' = Q$.

Let $p_i^w(b_i, b_{-i})$ denote the probability that bidder i wins when the bid vector is (b_i, b_{-i}). Define $W_i = \{t_{-i} \mid \max_{j\neq i} g_j^m(t_j) \le b_i\}$ as the set of types (a subset of T_{-i}) for which bidder i's bid b_i is the highest against the strategy profile $g_{-i}^m(\cdot)$, where $g^m(\cdot)$ is a monotone strategy profile. We define $\mathbb{E}(\cdot \mid t_i, W_i) = 0$ if $\mathbb{P}(W_i \mid t_i) = 0$. Then, we have the following:

$$\begin{aligned}
V_i(b_i, t_i; g_{-i}^m(\cdot)) &= \mathbb{P}(W_i|t_i)\mathbb{E}[w_i(b_i, t) p_i^w(b_i, g_{-i}^m(t_{-i}))|t_i, W_i] - b_i \\
&\le \mathbb{P}(W_i|t_i)\mathbb{E}[w_i(b_i, t)|t_i, W_i]\mathbb{E}[p_i^w(b_i, g_{-i}^m(t_{-i}))|t_i, W_i] - b_i \\
&\le \mathbb{P}(W_i|t_i)\mathbb{E}[w_i(b_i, t)|t_i, W_i] - b_i \\
&\le \lim_{b_i' \to b_i^+} V_i(b_i', t_i; g_{-i}^m),
\end{aligned} \tag{12}$$

where the first inequality follows from Milgrom and Weber (1982, Theorem 23), since both $w_i(b_i, t_i, t_{-i})$ and $1 - p_i^w(b_i, g_{-i}^m(t_{-i}))$ are increasing in t_{-i}. The second inequality holds because $0 \le \mathbb{E}[p_i^w(b_i, g_{-i}^m(t_{-i})) \mid t_i, W_i] \le 1$ and $\mathbb{P}(W_i \mid t_i)\mathbb{E}[w_i(b_i, t) \mid t_i, W_i] \ge 0$. The last equation is valid since (i) $p_i^w(b_i', g_{-i}^m(t_{-i})) = 1$ for all $b_i' > b_i$ and $t_{-i} \in W_i$; and (ii) for all

$t_{-i} \notin W_i$, $p_i^w(b_i', g_{-i}^m(t_{-i})) \to 0$ as $b_i' > b_i$ and $b_i' \to b_i$.

Notably, Inequality (12) also holds for any other monotone strategy profile $\psi(\cdot)$. Specifically, we have

$$V_i(b_i, t_i; \psi_{-i}(\cdot)) \leq \lim_{b_i' \to b_i^+} V_i(b_i', t_i; \psi_{-i}(\cdot)) \tag{13}$$

Let $\mathbb{P}(\gamma_{-i}) = \prod_{j \neq i} \mathbb{P}(\gamma_j)$ and $\mathbb{P}(\gamma_j) = (1 - \frac{1}{m})\delta_{g_j^m}(\gamma_j) + \frac{1}{2m}\delta_{U[\underline{b}_j, \bar{b}_j]}(\gamma_j) + \frac{1}{2m}\delta_{\delta_{\{Q\}}}(\gamma_j)$. We have

$$\begin{aligned}
V_i^m(b_i, t_i; g_{-i}^m(\cdot)) &= \sum_{\gamma_{-i} \in \prod_{j \neq i} \{g_j^m, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i}) V_i(b_i^m, t_i; \gamma_{-i}(\cdot)) \\
&= (1 - \frac{1}{m}) \sum_{\gamma_{-i} \in \prod_{j \neq i} \{g_j^m, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i}) V_i(b_i, t_i; \gamma_{-i}(\cdot)) \\
&\quad + \frac{1}{2m} \cdot \frac{1}{\bar{b}_i - \underline{b}_i} \sum_{\gamma_{-i} \in \prod_{j \neq i} \{g_j^m, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i}) \int_{[\underline{b}_i, \bar{b}_i]} V_i(\tilde{b}_i, t_i; \gamma_{-i}(\cdot))\, \mathrm{d}\tilde{b}_i \\
&\quad + 0
\end{aligned} \tag{14}$$

Given a strategy profile $\gamma_{-i} \in \prod_{j \neq i} \{g_j^m, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}$, we divide the players into three subsets. Let $I_1 = \{j \colon \gamma_j = U[\underline{b}_j, \bar{b}_j]\}$, which consists of bidders employing a uniform distribution strategy over their action sets. Define $I_2 = \{j \colon \gamma_j = \delta_{\{Q\}}\}$, which includes bidders with a degenerate strategy, placing all probability mass on Q. Let $I_3 = I \backslash (I_1 \cup I_2 \cup \{i\})$, which contains the remaining bidders. Then, by simple algebra, we obtain

$$\begin{aligned}
V_i(b_i, t_i; \gamma_{-i}(\cdot)) &= \int_{T_{-i}} u_i(b_i, t_i; \gamma_{-i}(t_{-i})) f(t_{-i}|t_i)\, \mathrm{d}t_{-i} \\
&= \prod_{j \in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{T_{-i}} \int_{\prod_{j \in I_1} [\underline{b}_j, \bar{b}_j]} u_i(b_i, t_i; b_{I_1}, Q_{I_2}, g_{I_3}^m) \bigotimes_{j \in I_1} \mathrm{d}b_j f(t_{-i}|t_i)\, \mathrm{d}t_{-i} \\
&= \prod_{j \in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{\prod_{j \in I_1} [\underline{b}_j, \bar{b}_j]} \int_{T_{-i}} u_i(b_i, t_i; b_{I_1}, Q_{I_2}, g_{I_3}^m) f(t_{-i}|t_i)\, \mathrm{d}t_{-i} \bigotimes_{j \in I_1} \mathrm{d}b_j \\
&= \prod_{j \in I_1} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{\prod_{j \in I_1} [\underline{b}_j, \bar{b}_j]} V_i(b_i, t_i; \delta_{b_{I_1}}, \delta_{Q_{I_2}}, g_{I_3}^m) \bigotimes_{j \in I_1} \mathrm{d}b_j,
\end{aligned}$$

where $b_{I_1} = (b_j)_{j \in I_1}$, $Q_{I_2} = (Q, \ldots, Q)$ with $\dim(Q_{I_2}) = \#|I_2|$, $\delta_{b_{I_1}} = (\delta_{\{b_j\}})_{j \in I_1}$, $\delta_{Q_{I_2}} = (\delta_{\{Q\}})_{j \in I_2}$, and $g_{I_3}^m = (g_j^m)_{j \in I_3}$. Since V_i is bounded, by the Lebesgue dominated convergence theorem and Inequality (13), we have

$$V_i(b_i, t_i; \gamma_{-i}) \leq \lim_{b_i' \to b_i^+} V_i(b_i', t_i; \gamma_{-i}). \tag{15}$$

Combining Equation (14) and Inequality (15), we obtain

$$V_i^m(b_i, t_i; g_{-i}^m(\cdot)) \leq \lim_{b_i' \to b_i^+} V_i^m(b_i', t_i; g_{-i}^m(\cdot)). \tag{16}$$

For each bidder j, because his strategy $g_j^m(t_j)$ has at most countably many mass points and A_j^k becomes dense in $[\underline{b}_j, \overline{b}_j] \cup \{Q\}$, thus, for every $b_i \in [\underline{b}_i, \overline{b}_i] \cup \{Q\}$, every $\epsilon > 0$, and λ_i almost all t_i, there exists $K \in \mathbb{N}$, and $\bar{b}_i \in A_i^K$ such that

$$\begin{aligned}\lim_{b_i' \to b_i^+} V_i^m(b_i', t_i; g_{-i}^m) &\leq V_i^m(\bar{b}_i, t_i; g_{-i}^m) + \epsilon \\ &\leq V_i^m(\bar{b}_i, t_i; g_{-i}^{mn_k}) + 2\epsilon \quad \text{for } k \geq K, \\ &\leq V_i^m(g_i^{mn_k}(t_i), t_i; g_{-i}^{mn_k}) + 2\epsilon \quad \text{for } k \geq K,\end{aligned} \tag{17}$$

where the first and second lines follow because $\bar{b}_i$ can be chosen such that the probability of any $g_j^m(t_j)$ equaling $\bar{b}_i$ is arbitrarily small. The third line follows because $\bar{b}_i \in A_i^K$ is a feasible action in G^{mn_k} for player i for every $k \geq K$, and g^{mn_k} is an equilibrium in the game G^{mn_k}.

Since $V_i(g_i^{mn_k}(t_i), t_i; g_{-i}^{mn_k}) = \mathbb{E}[w_i(g_i^{mn_k}(t_i), t_{-i}) p_i^w(g_i^{mn_k}(t_i), g_{-i}^{mn_k}(t_{-i})) \mid t_i] - g_i^{mn_k}(t_i)$, and since the probability that bidder i wins, $p_i^w(b_i, b_{-i})$, is increasing in b_i and decreasing in b_{-i}, each function $p_i^w(g^{mn_k}(t))$ in the sequence is monotone in each of its arguments $t_1, \ldots, t_n$: increasing in t_i and decreasing in t_{-i}. By Helly's selection theorem, extracting a subsequence if necessary, there exists a function $\eta_i\colon [0,1]^n \to [0,1]$ such that, for almost all $t \in [0,1]^n$, $\mathbb{E}[w_i(g_i^{mn_k}(t_i), t_{-i}) p_i^w(g_i^{mn_k}(t_i), g_{-i}^{mn_k}(t_{-i})) \mid t_i]$ converges to $\mathbb{E}[w_i(g_i^m(t_i), t_{-i}) \eta_i(t) \mid t_i]$, by the dominated convergence theorem. Since the bidders' finite action sets are pairwise disjoint, we have $p_i^w(g^{mn_k}(t)) \in \{0, 1\}$ for all n_k and t. Therefore, it follows that $\eta_i(t) \in \{0, 1\}$ for almost all t. One can interpret $\eta_i(\cdot)$ as a tie-breaking rule in the limit, with $\sum_{i=1}^n \eta_i \leq 1$ for almost all t.

Repeating the computation above, let $\mathbb{P}(\gamma_{-i}^{n_k}) = \prod_{j \neq i} \mathbb{P}(\gamma_j^{n_k})$, where $\mathbb{P}(\gamma_j^{n_k}) = \left(1 - \frac{1}{m}\right) \delta_{g_j^{mn_k}}(\gamma_j^{n_k}) + \frac{1}{2m} \delta_{U[0,1]}(\gamma_j^{n_k}) + \frac{1}{2m} \delta_{\delta_{\{Q\}}}(\gamma_j^{n_k})$. We obtain the following:

$$\begin{aligned}V_i^m(b_i, t_i; g_{-i}^{mn_k}(\cdot)) &= \sum_{\gamma_{-i}^{n_k} \in \prod_{j \neq i} \{g_j^{mn_k}, U[0,1], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i}^{n_k}) V_i(b_i^m, t_i; \gamma_{-i}^{n_k}(\cdot)) \\ &= (1 - \frac{1}{m}) \sum_{\gamma_{-i}^{n_k} \in \prod_{j \neq i} \{g_j^{mn_k}, U[0,1], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i}^{n_k}) V_i(b_i, t_i; \gamma_{-i}^{n_k}(\cdot)) \\ &\quad + \frac{1}{2m} \cdot \frac{1}{\bar{b}_i - \underline{b}_i} \sum_{\gamma_{-i} \in \prod_{j \neq i} \{g_j^{mn_k}, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i}^{n_k}) \int_{[\underline{b}_i, \bar{b}_i]} V_i(\tilde{b}_i, t_i; \gamma_{-i}^{n_k}(\cdot)) \, \mathrm{d}\tilde{b}_i \\ &\quad + 0\end{aligned} \tag{18}$$

Given a strategy profile $\gamma_{-i}^{n_k} \in \prod_{j \neq i} \{g_j^{mn_k}, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}$, we divide the players into three subsets. Let $I_1^{n_k} = \{j\colon \gamma_j^{n_k} = U[\underline{b}_j, \bar{b}_j]\}$, which consists of bidders employing a uniform distribution strategy over their action sets. Define $I_2^{n_k} = \{j\colon \gamma_j^{n_k} = \delta_{\{Q\}}\}$, which includes bidders with a degenerate strategy, placing all probability mass on Q. Let $I_3^{n_k} = I \backslash (I_1^{n_k} \cup I_2^{n_k} \cup \{i\})$, which contains the remaining bidders. Then, by simple algebra, we obtain

$$\begin{aligned}&V_i(b_i, t_i; \gamma_{-i}^{n_k}(\cdot)) \\ &= \int_{T_{-i}} u_i(b_i, t_i; \gamma_{-i}^{n_k}(t_{-i})) f(t_{-i} | t_i) \, \mathrm{d}t_{-i}\end{aligned} \tag{19}$$

$$= \prod_{j\in I_1^{n_k}} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{T_{-i}} \int_{\prod_{j\in I_1^{n_k}}[\underline{b}_j,\bar{b}_j]} u_i(b_i,t_i;b_{I_1^{n_k}},Q_{I_2^{n_k}},g^{mn_k}_{I_3^{n_k}}) \bigotimes_{j\in I_1^{n_k}} \mathrm{d}b_j f(t_{-i}|t_i)\,\mathrm{d}t_{-i}$$

$$= \prod_{j\in I_1^{n_k}} \frac{1}{\bar{b}_j - \underline{b}_j} \int_{T_{-i}} \int_{\prod_{j\in I_1^{n_k}}[\underline{b}_j,\bar{b}_j]} w_i(b_i,t_i,t_{-i})p_i^w(b_i,b_{I_1^{n_k}},Q_{I_2^{n_k}},g^{mn_k}_{I_3^{n_k}}) \bigotimes_{j\in I_1^{n_k}} \mathrm{d}b_j f(t_{-i}|t_i)\,\mathrm{d}t_{-i}$$
$$- b_i$$
$$= \int_{T_{-i}} w_i(b_i,t_i,t_{-i}) \prod_{j\in I_1^{n_k}} \frac{1}{\bar{b}_j - \underline{b}_j}\Big(\int_{\prod_{j\in I_1^{n_k}}[\underline{b}_j,\bar{b}_j]} p_i^w(b_i,b_{I_1^{n_k}},Q_{I_2^{n_k}},g^{mn_k}_{I_3^{n_k}}) \bigotimes_{j\in I_1^{n_k}} \mathrm{d}b_j\Big) f(t_{-i}|t_i)\,\mathrm{d}t_{-i}$$
$$- b_i, \tag{20}$$

where $b_{I_1^{n_k}} = (b_j)_{\{j\in I_1^{n_k}\}}$, $Q_{I_2^{n_k}} = (Q,\cdots,Q)$, with $\dim(Q_{I_2^{n_k}}) = \#|I_2^{n_k}|$, $\delta_{b_{I_1^{mn_k}}} = (\delta_{\{b_j\}})_{j\in I_1^{n_k}}$, $\delta_{Q_{I_2^{mn_k}}} = (\delta_{\{Q\}})_{j\in I_2^{n_k}}$, and $g^{mn_k}_{I_3^{n_k}} = (g_j^{mn_k})_{\{j\in I_3^{n_k}\}}$.

Denote $p_i^w(b_i,\gamma_{-i}^{n_k}) = \prod_{j\in I_1^{n_k}} \frac{1}{\bar{b}_j-\underline{b}_j}\int_{\prod_{j\in I_1^{n_k}}[\underline{b}_j,\bar{b}_j]} p_i^w(b_i,b_{I_1^{n_k}},Q_{I_2^{n_k}},g^{mn_k}_{I_3^{n_k}}(t_{I_3^{n_k}})) \otimes_{j\in I_1^{n_k}} \mathrm{d}a_j$. Since $p_i^w(\cdot)$ is decreasing in b_{-i}, we know that $p_i^w(b_i,\gamma_{-i}^{n_k})$ is decreasing in t_{-i}, and combine with Equation (19), we have

$$V_i(b_i,t_i;\gamma_{-i}^{n_k}(\cdot)) = \int_{T_{-i}} w_i(b_i,t_i,t_{-i})p_i^w(b_i,\gamma_{-i}^{n_k})f(t_{-i}|t_i)\,\mathrm{d}t_{-i} - b_i.$$

Similarly, we have $\{p_i^w(g_i^{mn_k}(t_i),\gamma_{-i}^{n_k})\}_{k=1}^\infty$ (extracting a subsequence if necessary), and a function $\eta_i^\gamma\colon [0,1]^n \to [0,1]$ such that for almost all $t\in[0,1]^n$,

$$\mathbb{E}[w_i(g_i^{mn_k}(t_i),t_{-i})p_i^w(g_i^{mn_k}(t_i),\gamma_{-i}^{mn_k}(t_{-i}))|t_i] \to \mathbb{E}[w_i(g_i^m(t_i),t_{-i})\eta_i^\gamma(t)|t_i]$$

by the dominated convergence theorem.

Denote $R_i^m(t_i,g_{-i}^{mn_k}) = \frac{1}{2m}\cdot\frac{1}{\bar{b}_i-\underline{b}_i}\sum_{\gamma_{-i}^{n_k}\in\prod_{j\neq i}\{g_j^{mn_k},U[\underline{b}_i,\bar{b}_i],\delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i}^{n_k})\int_{[\underline{b}_i,\bar{b}_i]} V_i(\tilde{b}_i,t_i;\gamma_{-i}^{n_k}(\cdot))\,\mathrm{d}\tilde{b}_i$. Define $\gamma_j = g_j^m$ if $\gamma_j^{n_k} = g_j^{mn_k}$ for all $k\in\mathbb{Z}_+$, and let $\gamma_j = \gamma_j^{n_k}$ if $\gamma_j^{n_k} = U[\underline{b}_i,\bar{b}_i]$ for all $k\in\mathbb{Z}_+$ or $\gamma_j^{n_k} = \delta_{\{Q\}}$ for all $k\in\mathbb{Z}_+$. Since the mass points of γ_{-i} and $\{\gamma_{-i}^{n_k}\}_{k\in\mathbb{N}}$ are at most countable, it follows that $R_i^m(t_i,g_{-i}^{mn_k})$ converges to $R_i^m(t_i,g_{-i}^m)$, where

$$R_i^m(t_i,g_{-i}^m) = \frac{1}{2m}\cdot\frac{1}{\bar{b}_i-\underline{b}_i}\sum_{\gamma_{-i}\in\prod_{j\neq i}\{g_j^m,U[\underline{b}_i,\bar{b}_i],\delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i})\int_{[\underline{b}_i,\bar{b}_i]} V_i(\tilde{b}_i,t_i;\gamma_{-i}(\cdot))\,\mathrm{d}\tilde{b}_i.$$

Since the set $\prod_{j\neq i}\{g_j^m,U[\underline{b}_j,\bar{b}_j],\delta_{\{Q\}}\}$ is finite, and $\epsilon>0$ is arbitrarily small, we have

$$\sup_{b_i\in[\underline{b}_i,\bar{b}_i]\cup\{Q\}} V_i^m(b_i,t_i;g_{-i}^m(\cdot))$$
$$\le \liminf_{k\to\infty} V_i^m(g_i^{mn_k}(t_i),t_i;g_{-i}^{mn_k}(\cdot))$$
$$= \lim_{l\to\infty} V_i^m(g_i^{mn_l}(t_i),t_i;g_{-i}^{mn_l}(\cdot))$$
$$= (1-\frac{1}{m})\sum_{\gamma_{-i}\in\prod_{j\neq i}\{g_j^m,U[0,1],\delta_{\{Q\}}\}} \mathbb{P}(\gamma_{-i})(\mathbb{E}[w_i(g_i^m(t_i),t_{-i})\eta_i^\gamma(t)|t_i] - g_i^m(t_i)) + R_i^m(t_i,g_{-i}^m), \tag{21}$$

where η_i^γ is the limit corresponding to the sequence $\{p_i^w(g_i^{mn_l}(t_i), \gamma_{-i}^{n_l})\}_{l=1}^\infty$. The first inequality follows from Inequalities (16) and (17). The first equation is obtained by selecting an appropriate subsequence of convergence, and the last equation follows from Equation (18) as $l \to \infty$.

Let $\hat{W}_i = \{t_{-i} | \max_{j\neq i} g_j^m(t_j) \leq g_i^m(t_i)\}$ be the set of types such that bidder i's bid $g_i^m(t_i)$ is the highest bid against $g^m(\cdot)$. Define $\mathbb{E}(\cdot \mid t_i, \hat{W}_i) = 0$ if $\mathbb{P}(\hat{W}_i \mid t_i) = 0$. Then we have

$$\begin{aligned}
&\mathbb{E}[w_i(g_i^m(t_i), t_{-i})\eta_i(t)|t_i] - g_i^m(t_i)\\
&= \mathbb{P}(\hat{W}_i|t_i)\mathbb{E}[w_i(g_i^m(t_i), t_{-i})\eta_i(t)|t_i, \hat{W}_i] - g_i^m(t_i)\\
&\leq \mathbb{P}(\hat{W}_i|t_i)\mathbb{E}[w_i(g_i^m(t_i), t_{-i})|t_i, \hat{W}_i]\mathbb{E}[\eta_i(t)|t_i, \hat{W}_i] - g_i^m(t_i)\\
&\leq \mathbb{P}(\hat{W}_i|t_i)\mathbb{E}[w_i(g_i^m(t_i), t_{-i})|t_i, \hat{W}_i] - g_i^m(t_i)\\
&= \lim_{\epsilon\to 0^+} V_i(g_i^m(t_i) + \epsilon, t_i; g_{-i}^m(\cdot)),
\end{aligned} \tag{22}$$

where the first inequality follows from Milgrom and Weber (1982, Theorem 23), and the second inequality follows from $0 \leq \mathbb{E}[\eta_i(t) \mid t_i, \hat{W}_i] \leq 1$ and $\mathbb{P}(\hat{W}_i \mid t_i)\mathbb{E}[w_i(g_i^m(t_i), t_{-i}) \mid t_i] \geq 0$.

Given a strategy profile $\gamma_{-i} \in \prod_{j\neq i}\{g_j^m, U[\underline{b}_j, \bar{b}_j], \delta_{\{Q\}}\}$. Let $I_1 = \{j : \gamma_j = U[\underline{b}_j, \bar{b}_j]\}$, which consists of bidders employing a uniform distribution strategy over their action sets. Let $\hat{W}_i^\gamma = \{t_{-i} \mid \max_{j\in I_3} g_j^m(t_j) \leq g_i^m(t_i)\}$. Define $\mathbb{E}(\cdot \mid t_i, \hat{W}_i^\gamma) = 0$ if $\mathbb{P}(\hat{W}_i^\gamma \mid t_i) = 0$. Then we have

$$\begin{aligned}
&\mathbb{E}[w_i(g_i^m(t_i), t_{-i})\eta_i^\gamma(t)|t_i] - g_i^m(t_i)\\
&= \mathbb{P}(\hat{W}_i^\gamma|t_i)\mathbb{E}[w_i(g_i^m(t_i), t_{-i})\eta_i^\gamma(t)|t_i, \hat{W}_i^\gamma] - g_i^m(t_i)\\
&\leq \mathbb{P}(\hat{W}_i^\gamma|t_i)\mathbb{E}[w_i(g_i^m(t_i), t_{-i})|t_i, \hat{W}_i^\gamma]\mathbb{E}[\eta_i^\gamma(t)|t_i, \hat{W}_i^\gamma] - g_i^m(t_i)\\
&\leq \mathbb{P}(\hat{W}_i^\gamma|t_i)\mathbb{E}[w_i(g_i^m(t_i), t_{-i})|t_i, \hat{W}_i^\gamma]\zeta^\gamma(t_i) - g_i^m(t_i)\\
&= \lim_{\epsilon\to 0^+} V_i(\hat{g}_i^m(t_i) + \epsilon, t_i; \gamma_{-i}),
\end{aligned} \tag{23}$$

where $\zeta^\gamma(t_i) = \prod_{j\in I_1} \max\{0, \min\{1, \frac{g_i^m(t_i)-\underline{b}_j}{\bar{b}_j-\underline{b}_j}\}\}$, the first inequality follows by Milgrom and Weber (1982, Thoerem 23), and the second inequality follows by $\mathbb{E}[\eta_i^\gamma(t)|t_i, \hat{W}_i^\gamma] \leq \zeta^\gamma(t_i)$ and $\mathbb{P}(\hat{W}_i^\gamma|t_i)\mathbb{E}[w_i(g_i^m(t_i), t_{-i})|t_i] \geq 0$. Hence, by Lebesgue dominated convergence theorem, we can obtain that

$$\begin{aligned}
&\lim_{l\to\infty} V_i^m(g_i^{mn_l}(t_i), t_i; g_{-i}^{mn_l})\\
&\leq \lim_{\epsilon\to 0^+} V_i^m(g_i^m(t_i) + \epsilon, t_i; g_{-i}^m)\\
&\leq \sup_{b_i\in[\underline{b}_i, \bar{b}_i]\cup\{Q\}} V_i^m(b_i, t_i; g_{-i}^m),
\end{aligned} \tag{24}$$

where the first inequality follows by Inequalities (21), (22) and (23).

Next, we show that the probability, under g^m, that two or more bidders simultaneously submit a highest bid above Q is 0. Combining Inequalities (21) and (24), we know that the inequalities in Inequalities (21) – (24) must be equalities. In particular, if $\mathbb{P}(\hat{W}_i|t_i) > 0$, then

$$0 \leq \mathbb{E}[w_i(g_i^m(t_i), t_{-i})|t_i, \hat{W}_i]\mathbb{E}[\eta_i(t)|t_i, \hat{W}_i] = \mathbb{E}[w_i(g_i^m(t_i), t_{-i})|t_i, \hat{W}_i].$$

Since $w_i(b_i, t_{-i}) > 0$ for all $b_i > Q$, and $t_{-i} \in [0,1]^{n-1}$, and $\mathbb{P}(\hat{W}_i|t_i) > 0$, this inequality strictly holds. Hence, we conclude that $\mathbb{E}[\eta_i(t)|t_i, \hat{W}_i] = 1$ for almost all t_i such that $\mathbb{P}(\hat{W}_i|t_i) > 0$.

Consequently, given $\emptyset \neq H \subseteq \{1, 2, \ldots, n\}$ and letting $T_H = \{t : g_i^m(t_i) = \max_j g_j^m(t_j) > Q, \forall i \in H\}$, we consider the probability $\mathbb{P}(T_H)$. If $\mathbb{P}(T_H) > 0$, then for every $i \in H$, $\eta_i(t) = 1$ for almost all $t \in T_H$. However, since $\sum_{i=1}^n \eta_i(t) \leq 1$ for almost all $t \in T$, we conclude that $|H| = 1$. Therefore, the probability that two or more bidders simultaneously submit the highest bid above Q under g^m is 0. Thus, for every i and almost all t_i, the function $V_i^m(\cdot, t_i; \cdot)$ is continuous at $(g_i^m(t_i), g_{-i}^m(t_{-i}))$. Therefore, we have $\lim_{l\to\infty} V_i^m(g_i^{mn_l}(t_i), t_i; g_{-i}^{mn_l}) = V_i^m(g_i^m(t_i), t_i; g_{-i}^m)$ for almost all t_i. This implies that $V_i^m(g_i^m(t_i), t_i; g_{-i}^m) = \sup_{b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}} V_i^m(b_i, t_i; \hat{g}_{-i}^m)$ for almost all t_i. Hence, g^m is a monotone equilibrium.

Step 4. By Helly's selection theorem, there exists a subsequence $\{g^{m_k}\}_{k=1}^\infty$ of the sequence $\{g^m\}_{m=1}^\infty$ that converges to a strategy profile g for almost all types t. In this step, we will show that g is a perfect monotone equilibrium.

By the construction of g, we have:

$$\lim_{k\to\infty} \rho(g_i^m(t_i), g_i(t_i)) = 0 \quad \text{for all } i, \quad \text{almost all } t_i.$$

Let $\overline{g}_j^m = (1 - \frac{1}{m})g_j^m + \frac{1}{2m}U[0,1] + \frac{1}{2m}\delta_{\{Q\}}$ for all j. Since g^m is an equilibrium in G^m, for almost all t_i, we have:

$$V_i^m(b_i, t_i; g_{-i}^m) \leq V_i^m(g_i(t_i); g_{-i}^m) \quad \text{for all } b_i \in [\underline{b}_j, \bar{b}_j] \cup \{Q\},$$

which is equivalent to:

$$V_i(b_i, t_i; \overline{g}_{-i}^m) \leq V_i(g_i(t_i), t_i; \overline{g}_{-i}^m) \quad \text{for all } b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}.$$

Thus, $g_i^m(t_i) \in \mathrm{BR}_i(t_i, \overline{g}_{-i}^m(t_i))$ for almost all t_i. Hence, the completely mixed strategy $\overline{g}^m$ in G satisfies:

$$\lim_{m\to\infty} \rho\left(\overline{g}^m(t_i), \mathrm{BR}_i(t_i, \overline{g}_{-i}^m(t_i))\right) = \lim_{m\to\infty} \rho\left(\overline{g}_i^m(t_i), g_i^m(t_i)\right) = 0.$$

To show that g is a perfect monotone equilibrium, we must show that g is an equilibrium. Note that each g^m is an equilibrium in G^m, and the game G^m converges to game G, and g^m converges to g. Thus, it remains to show that the probability of two or more players simultaneously submitting the highest bid above Q under g is zero. This can be derived using similar arguments to those in Step 3. We outline the key steps below.

From Inequality (13), we know that:

$$V_i(b_i, t_i; g(\cdot)) \leq \lim_{b_i' \to b_i^+} V_i(b_i', t_i; g(\cdot)).$$

We can obtain the following inequality (modified from Inequality (17)):

$$\begin{aligned}
\lim_{b_i' \to b_i^+} V_i(b_i', t_i; g_{-i}) &\leq V_i(\bar{b}_i, t_i; g_{-i}) + \epsilon \\
&\leq V_i(\bar{b}_i, t_i; g_{-i}^{m_k}) + 2\epsilon \quad \text{for } k \geq K \\
&\leq V_i(g_i^{m_k}(t_i), t_i; g_{-i}^{m_k}) + 3\epsilon \quad \text{for } k \geq K,
\end{aligned}$$

where the first and second inequalities hold for some $\bar{b}_i$ sufficiently close to b_i, and $\bar{b}_i$ is not a mass point for g and g^m for all m. The last inequality holds because g^{m_k} is an equilibrium in G^{m_k}, i.e., $V_i^m(\bar{b}_i, t_i; g_{-i}^{m_k}) \leq V_i^m(g_i^{m_k}(t_i), t_i; g_{-i}^{m_k})$. When k is sufficiently large, we can obtain that:

$$V_i(\bar{b}_i, t_i; g_{-i}^{m_k}) \leq V_i(g_i^{m_k}(t_i), t_i; g_{-i}^{m_k}) + \epsilon.$$

Thus, combining the above inequalities, we get

$$\sup_{b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}} V_i(b_i, t_i; g(\cdot)) \leq \lim_{k \to \infty} V_i(g_i^{m_k}(t_i), t_i; g_{-i}^{m_k}).$$

Notice that $V_i(g_i^{m_k}(t_i), t_i; g_{-i}^{m_k}) = \mathbb{E}[\omega_i(g_i^{m_k}, t_{-i}) p_i^w(g^{m_k})|t_i] - g_i^{m_k}(t_i)$. By the monotonicity of g and Helly's selection theorem, we can obtain $\mathbb{E}[\omega_i(g_i^{m_l}, t_{-i}) p_i^w(g^{m_l})|t_i] \to \mathbb{E}[\omega_i(g_i, t_{-i}) \eta_i^* | t_i]$ $\{n_l\}_{l=1}^{\infty}$ is a proper subsequence of $\{n_k\}_{k=1}^{\infty}$, by the dominated convergence theorem, where $\eta_i^* \colon [0,1]^n \to [0,1]$.

By the same arguments as the proof of Inequality (22), we can show that

$$\lim_{l \to \infty} V_i(g_i^{m_l}(t_i), t_i; g_{-i}^{m_l}) \leq \sup_{b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}} V_i(b_i, t_i; g(\cdot)).$$

Finally, we repeat the proof from Step 3, and we can show that the probability of two or more players simultaneously submitting the highest bid above Q under g is zero.

Thus, we have

$$\lim_{l \to \infty} V_i(g_i^{m_l}(t_i), t_i; g_{-i}^{m_l}) = V_i(g_i(t_i), t_i; g_{-i}).$$

Combining this with

$$\lim_{l \to \infty} V_i(g_i^{m_l}(t_i), t_i; g_{-i}^{m_l}) = \sup_{b_i \in [\underline{b}_i, \bar{b}_i] \cup \{Q\}} V_i(b_i, t_i; g(\cdot)),$$

we conclude that g is a monotone equilibrium. This completes the proof.

8.4 Proof of Theorem 2

In this section, the game G^m, each player i's interim payoff V_i^m, an increasing strategy profile ϕ, and the completely mixed strategy ϕ_i^m are as defined in the proof of Theorem 1 in Section 8.1.

Recall that each player i's payoff function $u_i(a,t) = u_i(a, t_i)$ depends only on the action profile and her own type. Additionally, players' type distributions are independent, so $f(t) = f_1(t_1) \dots f_n(t_n)$. For each player i, let μ_i^m be the distribution in A_i induced by ϕ_i^m, meaning that $\mu_i^m(B) = \int_{T_i} \phi_i^m(t_i, B) f_i(\mathrm{d}t_i)$, for any $B \subseteq A_i$. Thus, by simple algebra, we obtain

$$\begin{aligned} V_i^m(a_i, t_i; \phi_{-i}(\cdot)) &= \int_{T_{-i}} \int_{A_{-i}} u_i(a_i^m, a_{-i}, t_i) \prod_{j \in I, j \neq i} \phi_j^m(t_j, \mathrm{d}a_j) \prod_{j \in I, j \neq i} f_j(\mathrm{d}t_j) \\ &= \int_{A_{-i}} u_i(a_i^m, a_{-i}, t_i) \prod_{j \neq i, j \in I} \mu_j^m(\mathrm{d}a_j). \end{aligned}$$

Claim 7. *For each player j, there exists an increasing strategy w_j^m, such that w_j^m and ϕ_j^m induce the same distribution μ_j^m in A_j.*

Proof. Given a distribution μ_j^m on A_j. Let F_j^m denotes the cumulative distribution function on A_j such that $F_j^m(x) = \mu_j^m([\underline{a}_j, x])$.

Consider a sequence of monotone functions $\psi_j^k : T_j \to A_j$ defined by:

$$\psi_j^k = \sum_{q=1}^{k-1} a_j^{k,q} \delta_{[t_j^{k,q-1}, t_j^{k,q})} + a_j^{k,k} \delta_{[t_j^{k,k-1}, \bar{t}_j]},$$

where $a_j^{k,q} \in A_j$ is the minimum action such that $\mu_j^m([\underline{a}_j, a_j^{k,q}]) = \frac{q}{k}$, and $t_j^{k,q} \in [\underline{t}_j, \bar{t}_j]$ is the minimum type such that

$$\int_{[\underline{t}_j, t_j^{k,q}]} f_j(t_j)\, \mathrm{d}t_j = \frac{q}{k}, \text{ for each } q \in \{1, 2, \cdots, k\},$$

with $t_j^{k,0} = \underline{t}_j$. Since f_j is atomless, we know that ψ_j^k is well-defined. Therefore, ψ_j^k is a sequence of increasing functions, and the distribution induced by ψ_j^k converges weakly to μ_j^m. By Helly's selection theorem, we know that $\{\psi_j^k\}_{k=1}^\infty$ has a subsequence $\{\psi_j^{k_p}\}_{p=1}^\infty$ that converges almost everywhere, with the limit function denoted by w_j^m. Consequently, w_j^m is an increasing function mapping from T_j to A_j and the cumulative distribution function of w_j^m is F_j^m. Thus, we complete our proof. □

Hence, through simple algebra, we obtain:

$$\begin{aligned} V_i^m(a_i, t_i; \phi_{-i}(\cdot)) &= \int_{A_{-i}} u_i(a_i^m, a_{-i}, t_i) \prod_{j \neq i, j \in I} \mu_j^m(\mathrm{d}a_j) \\ &= \int_{T_{-i}} \int_{A_{-i}} u_i(a_i^m, a_{-i}, t_i) \prod_{j \in I, j \neq i} w_j^m(t_j, \mathrm{d}a_j) \prod_{j \in I, l \neq i} f_j(t_j)\, \mathrm{d}t_{-i} \\ &= V_i(a_i^m, t_i; w_{-i}^m(\cdot)). \end{aligned}$$

Moreover, $V_i^m(a_i, t_i; \phi_{-i}(\cdot)) - V_i^m(a_j, t_i; \phi_{-i}(\cdot)) = (1 - \frac{1}{m})(V_i(a_i, t_i; w_{-i}^m(\cdot)) - V_i(a_j, t_i; w_{-i}^m(\cdot)))$. Since the single crossing condition holds in game G, it also holds in game G^m. Thus, game G^m possesses a monotone equilibrium g^m for each $m \in \mathbb{N}$. By repeating Steps 2 and 3 of the proof of Theorem 1 in Section 8.1, we obtain the existence of perfect monotone equilibria in game G.

8.5 Proofs of Claims 1-3

Proof of Claim 1. 1. Since player 2's payoff function doesn't depend on any type, his interim payoff function $V_2(a_2, t_2; g_1(\cdot))$ satisfies $\mathrm{IDC}(a_2, t_2)$ for all $g_1 \in \mathcal{F}_1$, which implies that $V_2(a_2, t_2; g_1(\cdot))$ satisfies the single crossing condition in (a_2, t_2) ($\mathrm{SCC}(a_2, t_2)$) for all $g_1 \in \mathcal{F}_1$. Therefore, we only need to consider player 1's interim payoff function in this example. Given that player 2 plays an increasing strategy

$$s_2(t_2) = \begin{cases} 1 & t_2 \in [0, x_2) \\ 2 & t_2 \in [x_2, 1] \end{cases},$$

by simple algebra, we obtain that $V_1(2, t_1; s_2(\cdot)) - V_1(1, t_1; s_2(\cdot)) = (\frac{7}{4} - 2x_2)(1 + \frac{1}{6}x_2 - \frac{2}{3}t_1)$. Since $V_1(2, t_1; s_2(\cdot)) - V_1(1, t_1; s_2(\cdot))$ might not increasing in t_1, $V_1(a_1, t_1; s_2(\cdot))$

doesn't satisfy $\text{IDC}(a_1, t_1)$ for all $s_2 \in \mathcal{F}_2$. Additionally, since $1 + \frac{1}{6}x_2 - \frac{2}{3}t_1 > 0$ for all $t_1 \in [0,1]$, the sign of $V_1(2, t_1, s_2(\cdot)) - V_1(1, t_1, s_2(\cdot))$ doesn't depend on t_1. Thus, $V_1(a_1, t_1; s_2(\cdot))$ satisfies $\text{SCC}(a_1, t_1)$ for all $s_2 \in \mathcal{F}_2$.

2. Given that player 1 plays an increasing strategy

$$s_1(t_1) = \begin{cases} 1 & t_1 \in [0, x_1) \\ 2 & t_1 \in [x_1, 1] \end{cases},$$

by simple algebra, we have $V_2(2, t_2; s_1(\cdot)) - V_1(1, t_2; s_1(\cdot)) = 10x_1 - 8$. Since $V_1(2, t_1; s_2(\cdot)) - V_1(1, t_1; s_2(\cdot)) = (\frac{7}{4} - 2x_2)(1 + \frac{1}{6}x_2 - \frac{2}{3}t_1)$, we obtain

$$\text{BR}_1(x_2) = \begin{cases} x_1 = 0 & x_2 < \dfrac{7}{8} \\ x_1 \in [0,1] & x_2 = \dfrac{7}{8} \\ x_1 = 1 & x_2 > \dfrac{7}{8}, \end{cases} \qquad \text{BR}_2(x_1) = \begin{cases} x_2 = 1 & x_1 < \dfrac{4}{5} \\ x_2 \in [0,1] & x_1 = \dfrac{4}{5} \\ x_2 = 0 & x_1 > \dfrac{4}{5}. \end{cases}$$

Thus, the intersection of best response functions only has a unique point $(\frac{4}{5}, \frac{7}{8})$. It means that this game has a unique monotone equilibrium (s_1^*, s_2^*), where

$$s_1^*(t_1) = \begin{cases} 1 & t_1 \in [0, \dfrac{4}{5}) \\ 2 & t_1 \in [\dfrac{4}{5}, 1], \end{cases} \qquad s_2^*(t_2) = \begin{cases} 1 & t_2 \in [0, \dfrac{7}{8}) \\ 2 & t_2 \in [\dfrac{7}{8}, 1]. \end{cases}$$

3. To show this game does not possess any perfect monotone equilibrium is equivalent to proving that (s_1^*, s_2^*) is not a perfect equilibrium. We proceed by contradiction. Suppose (s_1^*, s_2^*) is a perfect equilibrium. Then, there exists a sequence of completely mixed strategies $\{(g_1^m, g_2^m)\}_{m=1}^{\infty}$ such that, for each player i, for almost all $t_i \in [0,1]$, the following conditions hold: (i) $\lim_{m\to\infty} \rho(g_i^m(t_i), s_i^*(t_i)) = 0$; (ii) $\lim_{m\to\infty} \rho(g_i^m(t_i), \text{BR}_i(t_i, g_{-i}^m)) = 0$. Let $\hat{g}_i^m(t_i) \in \text{BR}_i(t_i, g_{-i}^m)$ for all t_i be a measurable function. We will show that there exists $\overline{x}_i^m \in [0,1]$ such that

$$\hat{g}_i^m(t_i) = \begin{cases} 2 & t_i \in [0, \overline{x}_i^m) \\ 1 & t_i \in [\overline{x}_i^m, 1] \end{cases}.$$

By applying Helly's selection theorem, we know that there exists a subsequence of $\hat{g}_i^m(t_i)$ that converges pointwise to a decreasing function $\hat{g}_i^*$. Combining points (i) and (ii), we deduce that for almost all t_i, $\rho(\hat{g}_i^*(t_i), s_i^*(t_i)) = 0$. Since $\hat{g}_i^*$ is not an increasing function unless it is constantly equal to 2, and s_i^* is an increasing function that plays action 1 on a nonnegligible set, we can conclude that points (i) and (ii) cannot be satisfied simultaneously. This contradiction proves that the assumption that (s_1^*, s_2^*) is a perfect equilibrium is false. The detailed proof is presented below.

Let (s_1^*, s_2^*) be the unique monotone equilibrium, where

$$s_1^*(t_1) = \begin{cases} 1 & t_1 \in [0, \frac{4}{5}) \\ 2 & t_1 \in [\frac{4}{5}, 1], \end{cases} \qquad s_2^*(t_2) = \begin{cases} 1 & t_2 \in [0, \frac{7}{8}) \\ 2 & t_2 \in [\frac{7}{8}, 1]. \end{cases}$$

We will demonstrate that (s_1^*, s_2^*) is not a perfect equilibrium. To do so, consider a perturbation of player 2's strategy s_2^*, denoted as g_2^m, where

$$g_2^m(t_2) = \begin{cases} (1 - \phi^m(t_2))\delta_1 + \phi^m(t_2)\delta_2 & t_2 \in [0, \frac{7}{8}) \\ (1 - \phi^m(t_2))\delta_2 + \phi^m(t_2)\delta_1 & t_2 \in [\frac{7}{8}, 1]. \end{cases}$$

Similarly, a perturbation of player 1's strategy s_1^* can be defined as

$$g_1^m(t_1) = \begin{cases} (1 - \psi^m(t_2))\delta_1 + \psi^m(t_2)\delta_2 & t_1 \in [0, \frac{4}{5}) \\ (1 - \psi^m(t_2))\delta_2 + \psi^m(t_2)\delta_1 & t_1 \in [\frac{4}{5}, 1]. \end{cases}$$

Since $V_1(2, t_1; s_2^*) - V_1(1, t_1; s_2^*) = 0$. By simple algebra, we have

$$\begin{aligned}
& V_1(2, t_1; g_2^m) - V_1(1, t_1; g_2^m) \\
&= \int_0^{\frac{7}{8}} [u_1(2, 2, t_1, t_2) - u_1(2, 1, t_1, t_2)]\phi^m(t_2)\,\mathrm{d}t_2 \\
&\quad + \int_{\frac{7}{8}}^1 [u_1(2, 1, t_1, t_2) - u_1(2, 2, t_1, t_2)]\phi^m(t_2)\,\mathrm{d}t_2 \\
&\quad - \int_0^{\frac{7}{8}} [u_1(1, 2, t_1, t_2) - u_1(1, 1, t_1, t_2)]\phi^m(t_2)\,\mathrm{d}t_2 \\
&\quad - \int_{\frac{7}{8}}^1 [u_1(1, 1, t_1, t_2) - u_1(1, 2, t_1, t_2)]\phi^m(t_2)\,\mathrm{d}t_2 \\
&= \Big[\int_0^{\frac{7}{8}} (\frac{7}{6}t_2 - \frac{7}{24})\phi^m(t_2)\,\mathrm{d}t_2 - \int_{\frac{7}{8}}^1 (\frac{7}{6}t_2 - \frac{7}{24})\phi^m(t_2)\,\mathrm{d}t_2\Big] \\
&\quad - \Big[\int_0^{\frac{7}{8}} (-2 + \frac{1}{2}t_2 + \frac{4}{3}t_1)\phi^m(t_2)\,\mathrm{d}t_2 - \int_{\frac{7}{8}}^1 (-2 + \frac{1}{2}t_2 + \frac{4}{3}t_1)\phi^m(t_2)\,\mathrm{d}t_2\Big] \\
&= \int_0^{\frac{7}{8}} (\frac{2}{3}t_2 - \frac{4}{3}t_1 + \frac{41}{24})\phi^m(t_2)\,\mathrm{d}t_2 - \int_{\frac{7}{8}}^1 (\frac{2}{3}t_2 - \frac{4}{3}t_1 + \frac{41}{24})\phi^m(t_2)\,\mathrm{d}t_2 \\
&= (\frac{41}{24} - \frac{4}{3}t_1)\Big[\int_0^{\frac{7}{8}} \phi^m(t_2)\,\mathrm{d}t_2 - \int_{\frac{7}{8}}^1 \phi^m(t_2)\,\mathrm{d}t_2\Big] \\
&\quad + \frac{2}{3}\Big[\int_0^{\frac{7}{8}} t_2\phi^m(t_2)\,\mathrm{d}t_2 - \int_{\frac{7}{8}}^1 t_2\phi^m(t_2)\,\mathrm{d}t_2\Big].
\end{aligned}$$

Let $X^m = \int_0^{\frac{7}{8}} \phi^m(t_2)\,\mathrm{d}t_2 - \int_{\frac{7}{8}}^1 \phi^m(t_2)\,\mathrm{d}t_2$, $Y^m = \int_0^{\frac{7}{8}} t_2\phi^m(t_2)\,\mathrm{d}t_2 - \int_{\frac{7}{8}}^1 t_2\phi^m(t_2)\,\mathrm{d}t_2$.

Since $Y^m < \int_0^{\frac{7}{8}} \frac{7}{8}\phi^m(t_2)\,\mathrm{d}t_2 - \int_{\frac{7}{8}}^{1} \frac{7}{8}\phi^m(t_2)\,\mathrm{d}t_2 = \frac{7}{8}X^m$, we conclude that X^m and Y^m can not be 0 at the same time, meaning that $V_1(2, t_1; g_2^m) - V_1(1, t_1; g_2^m)$ cannot be 0 for all $t_1 \in [0, 1]$. Moreover, if $X^m \leq 0$, then $Y^m < 0$. And hence $V_1(2, t_1; g_2^m) - V_1(1, t_1; g_2^m) < 0$ for all $t_1 \in [0, 1]$. Thus, $V_1(2, t_1; g_2^m) - V_1(1, t_1; g_2^m)$ cannot be a strict increasing function of t_1 such that it can start at a negative value at $t_1 = 0$ and achieve a positive value for some $t_1 \in [0, 1]$. In other words, $V_1(2, t_1; g_2^m) - V_1(1, t_1; g_2^m)$ might be negative for all $t_1 \in [0, 1]$ or a decreasing function in t_1. Let $\hat{g}_1^m$ be a measurable selection of $\mathrm{BR}_1(g_2^m)$. Since $\mathrm{BR}_1(t_1, g_2^m)$ has a unique element for almost all t_1, $\hat{g}_1^m$ is unique defined for almost all t_1. Then, there exists $\overline{x}_1^m \in [0, 1]$ such that

$$\hat{g}_1^m(t_1) = \begin{cases} 2 & t_1 \in [0, \overline{x}_1^m) \\ 1 & t_1 \in [\overline{x}_1^m, 1]. \end{cases}$$

Suppose (s_1^*, s_2^*) is a perfect equilibrium, then there exists a sequence of completely mixed strategies $\{g_2^m\}_{m=1}^{\infty}$(resp. $\{g_1^m\}_{m=1}^{\infty}$) such that for almost all $t_2 \in [0, 1]$(resp. $t_1 \in [0, 1]$), $\lim_{m\to\infty} \rho(g_2^m(t_2), s_2^*(t_2)) = 0$(resp. $\lim_{m\to\infty} \rho(g_1^m(t_1), s_1^*(t_1)) = 0$), and it equivalents to that for almost all $t_2 \in [0, 1]$(resp. $t_1 \in [0, 1]$), $\lim_{m\to\infty} \phi^m(t_2) = 0$(resp. $\lim_{m\to\infty} \psi^m(t_1) = 0$). Let $\hat{g}_1^m$ be the best response function of player 1 given that player 2 plays g_2^m. As we showed above, we know that there exists $\overline{x}_1^m$ such that

$$\hat{g}_1^m(t_1) = \begin{cases} 2 & t_1 \in [0, \overline{x}_1^m) \\ 1 & t_1 \in [\overline{x}_1^m, 1]. \end{cases}$$

By the definition of perfect equilibrium, we have $\lim_{m\to\infty} \rho(g_1^m(t_1), \hat{g}_1^m(t_1)) = 0$, for almost all $t_1 \in [0, 1]$. By Helly's selection theorem, we know there exits a subsequence $\{\hat{g}_1^{m_k}\}_{k\in\mathbb{N}}$ of $\{\hat{g}_1^m\}_{m\in\mathbb{N}}$ that converges pointwise to a decreasing function $\hat{g}_1^*$, where

$$\hat{g}_1^*(t_1) = \begin{cases} 2 & t_1 \in [0, \overline{x}_1^*) \\ 1 & t_1 \in [\overline{x}_1^*, 1]. \end{cases}$$

Thus, we have

$$\rho(s_1^*(t_1), \hat{g}_1^*(t_1)) \leq \lim_{k\to\infty} [\rho(s_1^*(t_1), g_1^{m_k}(t_1)) + \rho(g_1^{m_k}(t_1), \hat{g}_1^{m_k}(t_1)) + \rho(\hat{g}_1^{m_k}(t_1), \hat{g}_1^*(t_1))] = 0,$$

for almost all $t_1 \in [0, 1]$. However, s_1^* is an increasing function with playing actions 1 and 2 on nonnegligible sets, and $\hat{g}_1^*$ is a decreasing function. Hence, $\rho(s_1^*(t_1), \hat{g}_1^*(t_1)) = 0$ cannot hold for almost all $t_1 \in [0, 1]$, which leads to a contradiction.

□

Proof of Claim 2. Given that bidder 1 plays the strategy b_1, the expected payoff for bidder

2, who submits bid amount a_2 at his value v_2, is:

$$V_2(a_2, v_2; b_1(\cdot)) = \begin{cases} v_2 \frac{3}{10} \cdot \frac{1}{2} & a_2 = 0 \\ (v_2 - 1)\frac{3}{10} \cdot \frac{3}{2} & a_2 = 1 \\ (v_2 - 2)\frac{3}{10} \cdot 2 & a_2 = 2 \\ (v_2 - 3)\frac{4}{5} & a_2 = 3 \\ v_2 - a_2 & a_2 \in \{4, \cdots, 8\}. \end{cases}$$

By simple algebra, we have

$$\text{BR}_2(b_1, v_2) = \begin{cases} 3 & v_2 \in [7, 8) \\ \{3, 4\} & v_2 = 8. \end{cases}$$

Given that bidder 2 always submits the bid amount 3, bidder 1's optimal payoff will be 0 if his valuation $v_1 \leq 3$. For $v_1 \in [3, 5]$, if bidder 1 submits a bid below 3, his expected payoff will be 0. If he submits a bid of 3, his expected payoff will be $\frac{1}{2}(v_1 - 3) \geq 0$. If he submits a bid $a_1 > 3$, his expected payoff will be $v_1 - a_1$, which is at most $v_1 - 4$. Since $\frac{1}{2}(v_1 - 3) \geq v_1 - 4$ for $v_1 \in [3, 5]$, we conclude that $b_1(v_1) \in \text{BR}_1(b_2, v_1)$ for all $v_1 \in [0, 5]$. Thus, (b_1, b_2) is a monotone equilibrium.

Next, we will prove that (b_1, b_2) is perfect. For $i \in \{1, 2\}$, let $\{b_i^m\}_{m=3}^{\infty}$ be a sequence of completely mixed strategies of bidder i, where

$$b_1^m(v_1) = \begin{cases} (1 - \frac{1}{m})\delta_0 + \sum_{k=1}^{8} \frac{1}{8m}\delta_k & v_i \in [0, \frac{3}{2}) \\ (1 - \frac{1}{m})\delta_1 + \sum_{k \neq 1} \frac{1}{8m}\delta_k & v_i \in [\frac{3}{2}, 3) \\ (1 - \frac{1}{m})\delta_3 + \sum_{k \neq 3} \frac{1}{8m}\delta_k & v_i \in [3, 5], \end{cases}$$

and $b_2^m(v_2) \equiv (1 - \frac{1}{m})\delta_3 + \sum_{k \neq 3} \frac{1}{8m}\delta_k$. Thus, we have $\lim_{m \to \infty} \rho(b_i^m(v_i), b_i(v_i)) = 0$, for all v_i, for $i \in \{1, 2\}$. Given that bidder 1 plays the strategy b_1^m, the expected payoff for bidder 2, who submits bid a_2 at his value v_2, is:

$$V_2(a_2, v_2; b_1^m(\cdot)) = \begin{cases} v_2 \frac{1}{2}[(1 - \frac{1}{m})\frac{3}{10} + \frac{1}{8m} \cdot \frac{7}{10}] & a_2 = 0 \\ (v_2 - 1)\frac{3}{2}[(1 - \frac{1}{m})\frac{3}{10} + \frac{1}{8m} \cdot \frac{7}{10}] & a_2 = 1 \\ (v_2 - 2)2[(1 - \frac{1}{m})\frac{3}{10} + \frac{1}{8m} \cdot \frac{7}{10}] & a_2 = 2 \\ (v_2 - 3)(2[(1 - \frac{1}{m})\frac{3}{10} + \frac{1}{8m} \cdot \frac{7}{10}] + \frac{1}{2}[(1 - \frac{1}{m})\frac{2}{5} + \frac{1}{8m} \cdot \frac{3}{5}]) & a_2 = 3 \\ (v_2 - a_2)[(1 - \frac{5}{8m}) + \frac{2(a_2 - 4) + 1}{16m}] & a_2 \in \{4, \cdots, 8\}. \end{cases}$$

For any m sufficiently large, the best response correspondence for bidder 2 against b_1^m is:

$$\text{BR}_2(b_1^m, v_2) = \begin{cases} 3 & v_2 \in [7, 8) \\ 4 & v_2 = 8. \end{cases}$$

And given bidder 2 that plays the strategy b_2^m, the expected payoff for bidder 1, who submits bid a_1 at his value v_1, is:

$$V_1(a_1, v_1; b_2^m(\cdot)) = \begin{cases} \frac{v_1}{2}\frac{1}{8m} & a_1 = 0 \\ \frac{3(v_1-1)}{2}\frac{1}{8m} & a_1 = 1 \\ \frac{5(v_1-2)}{2}\frac{1}{8m} & a_1 = 2 \\ (v_1 - 3)[(1 - \frac{1}{m})\frac{1}{2} + \frac{3}{8m}] & a_1 = 3 \\ (v_1 - a_1)[(1 - \frac{5}{8m}) + \frac{2(a_2 - 4) + 1}{16m}] & a_1 \in \{4, \cdots, 8\}. \end{cases}$$

For any m sufficiently large, the best response correspondence for bidder 1 against b_2^m is:

$$\text{BR}_1(b_2^m, v_1) = \begin{cases} 0 & v_1 \in [0, \frac{3}{2}) \\ \{0, 1\} & v_1 = \frac{3}{2} \\ 1 & v_1 \in (\frac{3}{2}, 3] \\ 3 & v_1 \in (3, 5]. \end{cases}$$

Thus, we have $\lim_{m\to\infty} \rho(b_j^m(v_j), \text{BR}_j(b_i^m, v_j)) = 0$, for almost all v_j, for $j \in \{1, 2\}$, and hence (b_1, b_2) is a perfect monotone equilibrium. □

Proof of Claim 3. Given that bidder i plays the strategy b_i, the expected payoff for bidder j ($j \neq i$, $i, j \in \{1, 2\}$), who submits bid a_j at his value v_j, is:

$$V_j(a_j, v_j; b_i(\cdot)) = \begin{cases} 0 & a_j = 0 \\ (v_j - 1)\frac{1}{4} & a_j = 1 \\ (v_j - 1)\frac{1}{2} + \frac{(v_j - 2)}{4} & a_j = 2. \end{cases}$$

By simple algebra, we have

$$\text{BR}_j(b_i, v_j) = \begin{cases} \{0, 1\} & v_j = 1 \\ 1 & v_j \in (1, \frac{3}{2}) \\ \{1, 2\} & v_j = \frac{3}{2} \\ 2 & v_j \in (\frac{3}{2}, 2], \end{cases}$$

thus (b_1, b_2) is a monotone equilibrium.

For each $i \in \{1,2\}$, let $\{b_i^m\}_{m=3}^{\infty}$ be a sequence of completely mixed strategies of bidder i, where

$$b_i^m(v_i) = \begin{cases} (1-\frac{2}{m})\delta_1 + \frac{1}{m}\delta_0 + \frac{1}{m}\delta_2 & v_i \in [1, \frac{3}{2}) \\ (1-\frac{2}{m})\delta_2 + \frac{1}{m}\delta_0 + \frac{1}{m}\delta_1 & v_i \in [\frac{3}{2}, 2]. \end{cases}$$

Thus, we have $\lim_{m\to\infty} \rho(b_i^m(v_i), b_i(v_i)) = 0$, for almost all $v_i \in [1,2]$, for $i \in \{1,2\}$. Given that bidder i plays the strategy b_i^m, the expected payoff for bidder j ($j \neq i$, $i, j \in \{1,2\}$), who submits bid a_j at his value v_j, is:

$$V_j(a_j, v_j; b_i^m(\cdot)) = \begin{cases} \frac{v_j}{2}\frac{1}{m} & a_j = 0 \\ v_j\frac{1}{m} + \frac{(v_j-1)}{4}(1-\frac{1}{m}) & a_j = 1 \\ v_j\frac{1}{m} + \frac{(v_j-1)}{2}(1-\frac{1}{m}) + \frac{(v_j-2)}{4}(1-\frac{1}{m}) & a_j = 2. \end{cases}$$

For any $m \geq 3$, the best response correspondence for bidder j against b_i^m is:

$$\mathrm{BR}_j(b_i^m, v_j) = \begin{cases} 1 & v_j \in [1, \frac{3}{2}) \\ \{1,2\} & v_j = \frac{3}{2} \\ 2 & v_j \in (\frac{3}{2}, 2]. \end{cases}$$

Thus, we have $\lim_{m\to\infty} \rho(b_j^m(v_j), \mathrm{BR}_j(b_i^m, v_j)) = 0$, for almost all $v_j \in [0,1]$, for $j \in \{1,2\}$. It implies that (b_1, b_2) is a perfect monotone equilibrium. $\square$